SPECTRAL SYMMETRY IN II_1-FACTORS

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Abstract. A self-adjoint element in a finite AW*-factor is *spectrally symmetric*, if its spectral measure under the quasitrace is invariant under the change of variables $t \longmapsto -t$. We show that if $\mathcal{A}$ is an AW*-factor of type II_1, a self-djoint element in $\mathcal{A}$, without full support, has quasitrace zero, if and only if it can be written as a sum of at most three commuting spectrally symmetric elements.

Introduction

According to the Murray-von Neumann classification, finite von Neumann factors are either of type I_{fin}, or of type II_1. For the non-expert, the easiest way to understand this classification is by accepting the famous result of Murray and von Neumann (see [9]) which states that *every finite von Neumann factor $\mathcal{M}$ posesses a unique state-trace $\tau_{\mathcal{M}}$*. Upon accepting this result, the type of $\mathcal{M}$ is decided by so-called *dimension range*: $\mathcal{D}_{\mathcal{M}} = \{\tau_{\mathcal{M}}(P) : P \text{ projection in } \mathcal{M}\}$ as follows. If $\mathcal{D}_{\mathcal{M}}$ is finite, then $\mathcal{M}$ is of type I_{fin} (more explictly, in this case $\mathcal{D}_{\mathcal{M}} = \{\frac{k}{n} : k = 0, 1, \dots, n\}$ for some $n \in \mathbb{N}$, and $\mathcal{M} \simeq \mathrm{Mat}_n(\mathbb{C})$ – the algebra of $n \times n$ matrices). If $\mathcal{D}_{\mathcal{M}}$ is infinite, then $\mathcal{M}$ is of type II_1, and in fact one has $\mathcal{D}_{\mathcal{M}} = [0, 1]$. From this point of view, the factors of type II_1 are the ones that are interesting, one reason being the fact that, although all factors of type II_1 have the same dimension range, there are uncountably many non-isomorphic ones (by a celebrated result of Connes).

In this paper we deal with a very simple problem. We start with a von Neumann II_1-factor $\mathcal{M}$, a (self-adjoint) element $A \in \mathcal{M}$, and we wish to characterize the condition: $\tau_{\mathcal{M}}(A) = 0$. The main feature of the trace $\tau_{\mathcal{M}}$ is

$$\tau_{\mathcal{M}}(XY - YX) = 0, \ \forall X, Y \in \mathcal{M}, \tag{1}$$

so a sufficient condition for $\tau_{\mathcal{M}}(A) = 0$ is that A be expressed as a *sum of commutators*, i.e. of elements of the form $[X, Y] = XY - YX$ with $X, Y \in \mathcal{M}$. A remarkable result due to Fack and de la Harpe ([3]) states not only that this condition is sufficient, but if $A = A^*$ then A can be written as a sum of at most five commutators of the form $[X, X^*]$.

The aim of this paper is to characterize the condition $\tau_{\mathcal{M}}(A) = 0$ in a way that is "Hilbert space free." What we have in mind of course is the purely algebraic setting due to Kaplansky ([5]), who attempted to formalize the theory of von Neumann algebras without any use of pre-duals. What emerged from Kaplansky's work was the concept of AW*-algebras, which we recall below.

Key words and phrases. Spectral symmetry, AW*-algebras, trace, quasitrace.

Definition. A unital C*-algebra $\mathcal{A}$ is called an *AW*-algebra*, if for every non-empty set $\mathcal{X} \subset \mathcal{A}$, the left anihilator set $\mathbf{L}(\mathcal{X}) = \{A \in \mathcal{A} : AX = 0, \ \forall X \in \mathcal{X}\}$ is the principal right ideal generated by a projection $P \in \mathcal{A}$, that is, $\mathbf{L}(\mathcal{X}) = \mathcal{A}P$.

One can classify the finite AW*-factors into the types $\mathrm{I}_{\mathrm{fin}}$ and II_1, exactly as above, but using the following alternative result: *any finite AW*-factor $\mathcal{A}$ posesses a unique normalized quasitrace $q_{\mathcal{A}}$*. Recall that a *quasitrace* on a C*-algebra $\mathfrak{A}$ is a map $q : \mathfrak{A} \to \mathbb{C}$ with the following properties:

(i) if $A, B \in \mathfrak{A}$ are self-adjoint, then $q(A + iB) = q(A) + iq(B)$;
(ii) $q(AA^*) = q(A^*A) \geq 0$, $\forall A \in \mathfrak{A}$;
(iii) q is linear on all abelian C*-subalgebras of $\mathfrak{A}$,
(iv) there is a map $q_2 : \mathrm{Mat}_2(\mathcal{A}) \to \mathbb{C}$ with properties (i)-(iii), such that

$$q_2\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = q(A), \ \ \forall A \in \mathfrak{A}.$$

(The condition that q is *normalized* means that $q(I) = 1$.)

With this terminology, the dimension range of a finite AW*-factor is the set $\mathcal{D}_{\mathcal{A}} = \{q_{\mathcal{A}}(P) : P \text{ projection in } \mathcal{A}\}$, and the classification into the two types is eaxctly as above. As in the case of von Neumann factors, one can show that the AW*-factors of type $\mathrm{I}_{\mathrm{fin}}$ are again the matrix algebras $\mathrm{Mat}_n(\mathbb{C})$, $n \in \mathbb{N}$. The type II_1 case however is still mysterious. In fact, a longstanding problem in the theory of AW*-algebras is the following:

Kaplansky's Conjecture. *Every AW*-factor of type* II_1 *is a von Neumann factor.*

An equivalent formulation states that: *if $\mathcal{A}$ is an AW*-factor of type* II_1*, then the quasitrace $q_{\mathcal{A}}$ is linear* (so it is in fact a trace). Why does one restrict Kaplansky's Conjecture to the case of factors? On the one hand, as Dixmier had shown, there are examples of abelian AW*-algebras which are not von Neumann algebras. Such algebras are those that lack the existence of sufficiently many normal states. (The spectra of such algebras are precisely those *stonean* spaces that are not *hyperstonean*.) On the other hand however, one has the following.

Fact A. *If $\mathcal{B}$ is an abelian AW*-subalgebra of a finite AW*-factor $\mathcal{A}$, then $\mathcal{B}$ is a von Neumann algebra.*

(This is due to the fact that the restriction $\tau = q_{\mathcal{A}}\big|_{\mathcal{B}} : \mathcal{B} \to \mathbb{C}$ is normal and faithful.) Actually (see [12] for example), a bit more can be said, namely:

Fact B. *If $\mathcal{M}$ is an AW*-subalgebra of a finite AW*-factor $\mathcal{A}$, such that the restriction $q_{\mathcal{A}}\big|_{\mathcal{M}} \to \mathbb{C}$ is linear, then $\mathcal{M}$ is a von Neumann algebra.*

A remarkable result of Haagerup ([4]) states that *if $\mathcal{A}$ is an AW*-factor of type* II_1*, generated (as an AW*-algebra) by an exact C*-algebra, then $\mathcal{A}$ is a von Neumann algebra.* A simple application of this result gives:

Fact C. *Any AW*-factor of type* II_1 *contains a unital AW*-subalgebra $\mathcal{R}$ that is $*$-isomorphic to the hyperfinite von Neumann* II_1*-factor.*

(This can be proven exactly as in the von Neumann case, buliding up first a copy of the diadic UHF algebra, and taking the AW*-completion.)

Suppose now $\mathcal{A}$ is an AW*-factor of type II_1, and $A \in \mathcal{A}$ is a self-adjoint element with $q_{\mathcal{A}}(A) = 0$. In our search for a characterization of this condition, it is worth

pointing out that, in the von Neumann case, the elements of the form $XX^* - X^*X$ are the ones that are "certain to have trace zero," whereas in the AW*-factor setting, this is not known to be the case. The natural question that arises in connection with this observation is: *which self-adjoint elements in $\mathcal{A}$ are "certain to have quasitrace zero"?* Since the only subsitute for (1) is

$$q_{\mathcal{A}}(UBU^*) = q_{\mathcal{A}}(B), \ \ \forall\, B \in \mathcal{A},\, U \in \mathbf{U}(\mathcal{A}), \tag{2}$$

where $\mathbf{U}(\mathcal{A})$ denotes the unitary group of $\mathcal{A}$, our supply of such elements, can consist of those self-adjoint elements $B \in \mathcal{A}$, for which there exists a unitary $U \in \mathbf{U}(\mathcal{A})$ with $UBU^* = -B$. It turns out that one can go even beyond these elements, by considering those self-adjoint B's which are *spectrally symmetric in $\mathcal{A}$*. This notion will be made precise in Section 2, but roughly speaking it means that the positive eigenvalues are the same as the negative eigenvalues, with equal multiplicities (which are computed using the quasitraces of the spectral projections). Using this (still imprecise) terminology, the main result of this paper states that *a self-adjoint element A with $q_{\mathcal{A}}(A) = 0$ can be written, after a suitable matrix stabilization, as the sum of three commuting spectrally symmetric elements.*

The paper is organized as follows. In Section 1 we discuss a certain type of convergence for nets in AW*-factors of type II_1, which is adequate when dealing with abelian ones. Section 2 covers the basic properties of approximate unitary equivalence and spectral symmetry. Section 3 deals with a certain integration technique that is inspired from von Neumann's minimax trace formula (see [9] and [3]). Section 4 contains most of the technical results. The main results are containd in Section 5.

Parts of this paper overlap with the first author's PhD dissertation. The first author wishes to express his gratitude to his thesis advisor Gabriel Nagy, for essential contributions to this project.

1. Weak convergence

In this section we discuss a possible substitute for weak convergence in the AW*-setting. We begin by adopting the following terminology. Given an AW*-algebra $\mathcal{A}$, we call a subalgebra $\mathcal{M} \subset \mathcal{A}$ a *von Neumann subalgebra*, if

- $\mathcal{M}$ is an AW*-algebra of $\mathcal{A}$;
- $\mathcal{M}$ is a von Neumann algebra, i.e. $\mathcal{M}$ is a dual Banach space.

The starting point in our discussion is the observation that *AW*-subalgebras of von Neuman subalgebras are von Neumann subalgebras.*

Definition. Let $\mathcal{A}$ be an AW*-algebra. We say that a net $(A_\lambda)_{\lambda\in\Lambda} \subset \mathcal{A}$ is *weakly convergent in $\mathcal{A}$*, if if there exists a von Neumann subalgebra $\mathcal{M}$ of $\mathcal{A}$, such that

(i) there exists some $\lambda_{\mathcal{M}} \in \Lambda$, such that $A_\lambda \in \mathcal{M}$, $\forall\, \lambda \succ \lambda_{\mathcal{M}}$;
(iii) the net $(A_\lambda)_{\lambda\in\Lambda}$ is convergent in $\mathcal{M}$ in the $\mathrm{w}^*_{\mathcal{M}}$-topology.

Observe that in this case, if $\mathcal{N}$ is any other von Neumann subalgebra of $\mathcal{A}$ with property (i), then it will satisfy condition (ii) automatically. Indeed, if we choose $\mu \in \Lambda$ such that $\mu \succ \lambda_{\mathcal{M}}$ and $\mu \succ \lambda_{\mathcal{N}}$, then $A_\lambda \in \mathcal{M}\cap\mathcal{N}$, $\forall\, \lambda \succ \mu$. Moreover $\mathcal{M}\cap\mathcal{N}$ is a von Neumann subalgebra in both $\mathcal{M}$ and $\mathcal{N}$, so one will have the equalities

$$\mathrm{w}^*_{\mathcal{M}} \lim_{\lambda\in\Lambda} A_\lambda = \mathrm{w}^*_{\mathcal{N}} \lim_{\lambda\in\Lambda} A_\lambda = \mathrm{w}^*_{\mathcal{M}\cap\mathcal{N}} \lim_{\lambda\in\Lambda} A_\lambda \in \mathcal{M}\cap\mathcal{N}.$$

In particular, the limit $\mathrm{w}^*_{\mathcal{M}}\lim_{\lambda\in\Lambda} A_\lambda$ is independent on the particular choice of $\mathcal{M}$ – as long as $\mathcal{M}$ has properties (i)-(ii). This element will then be denoted by $\mathsf{W}\text{-}\lim_{\lambda\in\Lambda} A_\lambda$, and will be referred to as the weak limit of the net $(A_\lambda)_{\lambda\in\Lambda}$. (When there is any danger of confusion, the notation $\mathsf{W}_{\mathcal{A}}$-lim will be used.)

Remark 1.1. Assume $\mathcal{A}$ is an AW*-algebra, and $\mathcal{B}$ is an AW*-subalgebra of $\mathcal{A}$. For a net $(A_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}$, the conditions:

(i) $(A_\lambda)_{\lambda\in\Lambda}$ is weakly convergent in $\mathcal{A}$, and
(ii) $(A_\lambda)_{\lambda\in\Lambda}$ is weakly convergent in $\mathcal{B}$,

are equivalent, and moreover one has the equality $\mathsf{W}_{\mathcal{A}}\text{-}\lim_{\lambda\in\Lambda} A_\lambda = \mathsf{W}_{\mathcal{B}}\text{-}\lim_{\lambda\in\Lambda} A_\lambda$.

Indeed, if condition (i) is satisfied, there exists some von Neumann subalgebra $\mathcal{M} \subset \mathcal{A}$, such that $A_\lambda \in \mathcal{M}$, $\forall\,\lambda \succ \lambda_M$, and some element $A \in \mathcal{M}$, such that $\mathrm{w}^*_{\mathcal{M}}\lim_{\lambda\in\Lambda} A_\lambda = A$. In this case, we simply notice that $\mathcal{N} = \mathcal{M} \cap \mathcal{B}$ is a von Neumann subalgebra of $\mathcal{B}$ (hence also of $\mathcal{A}$), so by the above discussion we must have $\mathrm{w}^*_{\mathcal{N}}\lim_{\lambda\in\Lambda} A_\lambda = A$. The implication (ii)$\Rightarrow$ (i) is trivial, since any von Neumann subalgebra of $\mathcal{B}$ is also a von Neumann subalgebra of $\mathcal{A}$.

Comment. In what follows we are going to restrict ourselves to the case when the ambient AW*-algebra $\mathcal{A}$ is a *finite factor*. In this case the key observation is the fact that (see the introduction) if $\mathcal{M} \subset \mathcal{A}$ is an AW*-algebra with the property that the restriction $q_{\mathcal{A}}\big|_{\mathcal{M}} : \mathcal{M} \to \mathbb{C}$ is linear, then $\mathcal{M}$ is a von Neumann algebra. In particular, *all abelian AW*-subalgebras of* $\mathcal{A}$ *are von Neumann subalgebras.*

With the above discussion in mind, the following terminology will be useful.

Definitions. Let $\mathcal{A}$ be a $*$-algebra. For $\mathcal{X} \subset \mathcal{A}$, define $\mathcal{X}^* = \{X^* : X \in \mathcal{X}\}$.

(a) A subset $\mathcal{X} \subset \mathcal{A}$ is said to be *abelian* if $XY = YX$, $\forall\, X, Y \in \mathcal{X}$.
(b) A subset $\mathcal{X} \subset \mathcal{A}$ is said to be $*$-*abelian* if $\mathcal{X} \cup \mathcal{X}^*$ is abelian.
(c) A subset $\mathcal{X} \subset \mathcal{A}$ is said to be *involutive* if $\mathcal{X}^* = \mathcal{X}$.

It is obvious that, if $\mathcal{X}$ is involutive, then "$*$-abelian" is equivalent to "abelian." This is the case for instance when $\mathcal{X} \subset \mathcal{A}_{sa} (= \{A \in \mathcal{A} : A = A^*\})$.

Remark 1.2. If $\mathcal{A}$ is a finite AW*-factor, and if $\mathcal{X} \subset \mathcal{A}$ is a $*$-abelian subset, then $\mathcal{X}$ is contained in an abelian von Neumann subalgebra $\mathcal{M} \subset \mathcal{A}$, for example $\mathcal{M} = (\mathcal{X} \cup \mathcal{X}^*)''$ – the bicommutant of $\mathcal{X} \cup \mathcal{X}^*$ in $\mathcal{A}$.

Remark 1.3. The above observation is useful when dealing with $*$-abelian nets. More explicitly, if $\mathcal{A}$ is a finite AW*-factor, and $(A_\lambda)_{\lambda\in\Lambda} \subset \mathcal{A}$ is a $*$-abelian net – as a set $\mathcal{X} = \{A_\lambda : \lambda \in \Lambda\}$ – then the condition, that $(A_\lambda)_{\lambda\in\Lambda}$ is weakly convergent in $\mathcal{A}$, is equivalent to the condition that $(A_\lambda)_{\lambda\in\Lambda}$ is convergent in $\mathcal{M} = (\mathcal{X} \cup \mathcal{X}^*)''$ in the $\mathrm{w}^*_{\mathcal{M}}$-topology.

Remark 1.4. If $\mathcal{A}$ is a finite AW*-factor, the operation of taking weak limits of abelian nets in $\mathcal{A}$ is "quasi-linear," in the following sense. If $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ are weakly convergent *jointly* $*$-abelian nets in $\mathcal{A}$, meaning that the set

$$\mathcal{X} = \{A_\lambda : \lambda \in \Lambda\} \cup \{B_\lambda : \lambda \in \Lambda\})$$

is $*$-abelian, then for any $\zeta \in \mathbb{C}$, the (abelian) net $(A_\lambda + \zeta B_\lambda)_{\lambda\in\Lambda}$ is weakly convergent, with limit

$$\mathop{\mathsf{W}\text{-}\lim}_{\lambda\in\Lambda}(A_\lambda + \zeta B_\lambda) = \big[\mathop{\mathsf{W}\text{-}\lim}_{\lambda\in\Lambda} A_\lambda\big] + \zeta\big[\mathop{\mathsf{W}\text{-}\lim}_{\lambda\in\Lambda} A_\lambda\big].$$

Lemma 1.1. *Let $\mathcal{A}$ be a finite AW*-factor, and let $(A_\lambda)_{\lambda\in\Lambda}$ be an abelian net in $\mathcal{A}_{sa}$, which is*

- *bounded, i.e.* $\sup_{\lambda\in\Lambda}\|A_\lambda\| < \infty$, *and*
- *monotone, i.e. has one of the properties* $(\uparrow)$ *or* $(\downarrow)$ *below:*
  $(\uparrow)$ $\lambda_1 \succ \lambda_2 \Rightarrow A_{\lambda_1} \geq A_{\lambda_2}$,
  $(\downarrow)$ $\lambda_1 \succ \lambda_2 \Rightarrow A_{\lambda_1} \leq A_{\lambda_2}$.

Then the net $(A_\lambda)_{\lambda\in\Lambda}$ is weakly convergent. Moreover, if we take $A = \mathsf{W}\text{-}\lim_{\lambda\in\Lambda} A_\lambda$, then for any integer $k \geq 1$, one has the following properties:

(i) *the net $(A_\lambda^k)_{\lambda\in\Lambda}$ is weakly convergent, and* $\mathsf{W}\text{-}\lim_{\lambda\in\Lambda} A_\lambda^k = A^k$;
(ii) $q_{\mathcal{A}}(A^k) = \lim_{\lambda\in\Lambda} q_{\mathcal{A}}(A_\lambda^k)$.

Proof. Consider the bicommutant $\mathcal{M} = \{A_\lambda : \lambda \in \Lambda\}''$, which is a von Neumann algebra. By Remark 1.3, in order to prove the first statement, and statement (i), it suffices to show that: the nets $(A_\lambda^k)_{\lambda\in\Lambda}$, $k \in \mathbb{N}$ are all $\mathrm{w}^*_{\mathcal{M}}$ convergent, and moreover, $\mathrm{w}^*_{\mathcal{M}} \lim_{\lambda\in\Lambda} A_\lambda = A^k$, $\forall\, k \in \mathbb{N}$, where $A = \mathrm{w}^*_{\mathcal{M}} \lim_{\lambda\in\Lambda} A_\lambda$. The fact that the nets $(A_\lambda^k)_{\lambda\in\Lambda}$, $k \in \mathbb{N}$ are convergent is clear, since all these nets are monotone and bounded (the fact that $(A_\lambda)_{\lambda\in\Lambda}$ is abelian is key for the monotonicity). To prove the second assertion, we define $X_k = \mathrm{w}^*_{\mathcal{M}} \lim_{\lambda\in\Lambda} A_\lambda^k$, and we notice that, due to the monotonicity and boundedness of the nets $(A_\lambda^k)_{\lambda\in\Lambda}$, we actually have:

$$X_k = \textsc{so-}\lim_{\lambda\in\Lambda} A_\lambda^k \text{ (in } \mathcal{M}),$$

where "$\textsc{so}$" stands for the *strong operator topology*, (coming from a realization of $\mathcal{M}$ as a von Neumann algebra on some Hilbert space). Since $(A_\lambda)_{\lambda\in\Lambda}$ is bounded, this gives

$$\textsc{so-}\lim_{\lambda\in\Lambda} A_\lambda^k = A^k \text{ (in } \mathcal{M}),$$

so we indeed have the equalities $X_k = A^k$. Finally, since $q_{\mathcal{A}}\big|_{\mathcal{M}}$ is a normal linear functional, it follows that

$$\lim_{\lambda\in\Lambda} q_{\mathcal{A}}(A_\lambda^k) = q_{\mathcal{A}}(A^k), \ \ \forall\, k \geq 1. \quad \square$$

2. Approximate Unitary Equivalence and Spectral Symmetry

Notations. Let $\mathcal{A}$ be a unital C*-algebra.
A. Two elements $A, B \in \mathcal{A}$ are said to be *orthogonal*, in which case we write $A \perp B$, if: $AB = BA = AB^* = B^*A = 0$. (Using the Fuglede-Putnam Theorem, in the case when one of the two is normal, the above condition reduces to: $AB = BA = 0$. If both A and B are normal, one only needs $AB = 0$.) A collection $(A_j)_{j\in J} \subset \mathcal{A}$ is said to be orthogonal, if $A_i \perp A_j$, $\forall\, i \neq j$.
B. We denote by $\mathbf{U}(\mathcal{A})$ the group of unitaries in $\mathcal{A}$. We denote by $\mathbf{P}(\mathcal{A})$ the collection of projections in $\mathcal{A}$, that is, $\mathbf{P}(\mathcal{A}) = \{P \in \mathcal{A}_{sa} : P = P^2\}$.

Definition. Let $\mathcal{A}$ be a unital C*-algebra. Two elements $A, B \in \mathcal{A}$ are said to be *approximately unitarily equivalent*, if there exists a sequence $(U_n)_{n=1}^\infty \subset \mathbf{U}(\mathcal{A})$ such that $\lim_{n\to\infty}\|U_n A U_n^* - B\| = 0$. In this case we write $A \sim B$.

The following result (perhaps well known) collects several easy properties of $\sim$.

Proposition 2.1. *Let $\mathcal{A}$ be a unital C*-algebra.*

(i) *The relation $\sim$ is an equivalence relation on $\mathcal{A}$.*

(ii) *If $A = \lim_{n\to\infty} A_n$ and $B = \lim_{n\to\infty} B_n$ (in norm), and if $A_n \sim B_n$, $\forall\, n$, then $A \sim B$.*
(iii) *If $A, B \in \mathcal{A}$ are such that $A \sim B$, then A and B have the same norm and the same spectrum.*
(iv) *If $A, B \in \mathcal{A}$ are such that $A \sim B$, then $A^* \sim B^*$.*
(v) *If $A, B \in \mathcal{A}$ are such that $A \sim B$ and A is normal, then B is also normal, and furthermore $f(A) \sim f(B)$ for any continuous function $f : \mathbb{C} \to \mathbb{C}$.*

Proof. (i). The reflexivity is trivial. The symmetry is clear because of the equality

$$\|UAU^* - B\| = \|U(A - U^*BU)U^*\| = \|A - U^*BU\|, \ \ \forall\, A, B \in \mathcal{A},\, U \in \mathbf{U}(\mathcal{A}).$$

The transitivity is a consequence of the inequality:

$$\begin{aligned}\|VUAU^*V^* - C\| &\leq \|V(UAU^* - B)V^*\| + \|VBV^* - C\| \\ &= \|UAU^* - B\| + \|VBV^* - C\|, \ \ \forall\, A, B, C \in \mathcal{A},\, U, V \in \mathbf{U}(\mathcal{A}).\end{aligned}$$

(ii). This is pretty clear, since for every unitary $U \in \mathbf{U}(\mathcal{A})$ one has the inequalities

$$\|UAU^* - B\| \leq \|A - A_n\| + \|B - B_n\| + \|UA_nU^* - B_n\|,$$

so if we choose, for each n, a unitary $U_n \in \mathbf{U}(\mathcal{A})$, such that $\|U_nA_nU_n^* - B_n\| < \frac{1}{n}$, then $\lim_{n\to\infty} U_nAU_n^* = B$ (in norm).

(iii). Assume $A \sim B$. The equality $\|A\| = \|B\|$ is obvious, since $\|UAU^*\| = \|A\|$, $\forall\, U \in \mathbf{U}(\mathcal{A})$. To prove that A and B have the same spectrum, it suffices (by symmetry) to prove that, for every $\lambda \in \mathbb{C}$, one has the implication: $A - \lambda I$ invertible $\Rightarrow$ $B - \lambda I$ invertible. If we choose $(U_n)_{n=0}^\infty \subset \mathbf{U}(\mathcal{A})$, with $\lim_{n\to\infty} \|U_nAU_n^* - B\| = 0$, then it is trivial that

$$\lim_{n\to\infty} \|U_n(A - \lambda I)U_n^* - (B - \lambda I)\| = 0,$$

so $B - \lambda I$ is the (norm) limit of a sequence $X_n = U_n(A - \lambda I)U_n^*$, $n \geq 0$, whose terms are all invertible elements. Since $\|X_n^{-1}\| = \|(A - \lambda I)^{-1}\|$, $\forall\, n \geq 0$, we get

$$\lim_{n\to\infty} \|I - X_n^{-1}(B - \lambda I)\| = \lim_{n\to\infty} \|I - (B - \lambda I)X_n^{-1}\| = 0,$$

so for n large both $X_n^{-1}(B - \lambda I)$ and $(B - \lambda I)X_n^{-1}$ are invertible, and so is $B - \lambda I$.

(iv). This is trivial, since

$$\|UA^*U^* - B^*\| = \|(UAU^* - B)^*\| = \|UAU^* - B\|, \ \ \forall\, A, B \in \mathcal{A},\, U \in \mathbf{U}(\mathcal{A}).$$

(v). Assume A is normal, and $A \sim B$. If we choose a sequence of unitaries $(U_n)_{n=0}^\infty \subset \mathbf{U}(\mathcal{A})$ with $B = \lim_{n\to\infty} U_nAU_n^*$ (in norm), then by (iv) we also have $B^* = \lim_{n\to\infty} U_nA^*U_n^*$ (in norm), so we get the equalities

$$\begin{aligned}BB^* &= \lim_{n\to\infty} (U_nAU_n^*)(U_nA^*U_n^*) = \lim_{n\to\infty} U_nAA^*U_n^*, \\ B^*B &= \lim_{n\to\infty} (U_nA^*U_n^*)(U_nAU_n^*) = \lim_{n\to\infty} U_nA^*AU_n^*,\end{aligned}$$

(in norm) so we clearly have $BB^* = B^*B$. Notice now that we also have

$$B^kB^{*\ell} = \lim_{n\to\infty} U_nA^kA^{*\ell}U_n^*, \ \ \forall\, k, \ell \geq 0,$$

so in fact we get

$$p(B, B^*) = \lim_{n\to\infty} U_np(A, A^*)U_n^*,$$

for every polynomial $p(t, s)$ of two variables. Using the Stone-Weierstrass Theorem, one then immediately gets $f(B) = \lim_{n\to\infty} U_nf(A)U_n^*$, for every continuous

function $f : K \to \mathbb{C}$, where K denotes the spectrum of A (which is the same as the spectrum of B). $\square$

Below we take a closer look at approximate unitary equivalence, in the case when ambient C*-algebra is a finite AW*-factor. To make matters a bit simpler, we restrict our attention to self-adjoint elements.

Notation. Let $\mathcal{A}$ be a finite AW*-factor. The restriction of the quasitrace $q_{\mathcal{A}}$ to $\mathbf{P}(\mathcal{A})$ will be denoted by $D_{\mathcal{A}}$ (or simply D, when there is no danger of confusion). The map $D : \mathbf{P}(\mathcal{A}) \to [0, 1]$ is referred to as the *dimension function* on $\mathcal{A}$.

For future reference, we collect the important properties of the dimension function, in the following.

Proposition 2.2. *Let $\mathcal{A}$ be a finite AW*-factor.*

(i) *For $P, Q \in \mathbf{P}(\mathcal{A})$, the following are equivalent:*
  - *$P \sim Q$;*
  - *there exists $U \in \mathbf{U}(\mathcal{A})$, such that $UPU^* = Q$;*
  - *there exists $V \in \mathcal{A}$ with $VV^* = P$ and $V^*V = Q$;*
  - *$D(P) = D(Q)$.*

(ii) *If a collection $(P_j)_{j\in J} \subset \mathbf{P}(\mathcal{A})$ is orthogonal, then*
$$D\Big(\bigvee_{j\in J} P_j\Big) = \sum_{j\in J} D(P_j).$$

Proof. See [5]. $\square$

Notations. Let $\mathcal{A}$ be a finite AW*-factor.
A. For an element $A \in \mathcal{A}_{sa}$, we denote by $\mu^A_{\mathcal{A}}$ the spectral measure of A under the quasitrace $q_{\mathcal{A}}$. (If there is no danger of confusion, we are going to omit the subscript $\mathcal{A}$ from the notation.) To define rigourously μ^A, we have to consider the space $C_0(\mathbb{R})$ of all continuous complex-valued functions on $\mathbb{R}$, which vanish at $\pm\infty$, and we use Riesz' Theorem to define μ^A to be the unique (probability) measure on $Bor(\mathbb{R})$ – the Borel σ-algebra – which satisfies the equality
$$q_{\mathcal{A}}\big(f(A)\big) = \int_{\mathbb{R}} f\, d\mu^A,\ \ \forall\, f \in C_0(\mathbb{R}).$$
The measure μ^A will be called the *scalar spectral measure of A, relative to $\mathcal{A}$.*
B. Given an element $A \in \mathcal{A}_{sa}$, its bicommutant $\{A\}''$ is an abelian von Neumann algebra (by the discussion in Section 1). For any Borel set $B \subset \mathbb{R}$ we denote by $e_B : \mathbb{R} \to \mathbb{R}$ its indicator function, and then using Borel functional calculus in $\{A\}''$ we can construct a projection, denoted $e_B(A) \in \{A\}''$. By construction, one has the equality
$$D\big(e_B(A)\big) = \mu^A(B),\ \ \forall\, B \in Bor(\mathbb{R}). \tag{3}$$

With these notations, one has the following result.

Theorem 2.1. *Let $\mathcal{A}$ be a finite AW*-factor. For two elements $A, B \in \mathcal{A}_{sa}$, the following are equivalent:*

(i) *$A \sim B$;*
(ii) *$\mu^A = \mu^B$, as measures on $Bor(\mathbb{R})$;*
(iii) *$D\big(e_{(-\infty,\lambda)}(A)\big) = D\big(e_{(-\infty,\lambda)}(B)\big)$, $\forall\, \lambda \in \mathbb{R}$;*
(iv) *$D\big(e_{(-\infty,\lambda]}(A)\big) = D\big(e_{(-\infty,\lambda]}(B)\big)$, $\forall\, \lambda \in \mathbb{R}$;*

(v) $q_{\mathcal{A}}\big(f(A)\big) = q_{\mathcal{A}}\big(f(B)\big)$, *for every continuous function* $f : \mathbb{R} \to \mathbb{C}$*;*
(vi) $q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(B^k)$, $\forall\, k \in \mathbb{N}$.

Proof. (i)⇒ (vi). By Proposition 2.1, it suffices to consider the case $k = 1$. Assume $A \sim B$, so there exists $(U_n)_{n=0}^{\infty} \subset \mathbf{U}(\mathcal{A})$ such that $B = \lim_{n\to\infty} U_n A U_n^*$ (in norm). Since the quasitrace is norm continuous (see [2]), the equality $q_{\mathcal{A}}(A) = q_{\mathcal{A}}(B)$ follows immediately from (2).

(vi)⇒ (v). Assume (vi), fix a continuous function $f : \mathbb{R} \to \mathbb{C}$, and let us prove the equality

$$q_{\mathcal{A}}\big(f(A)\big) = q_{\mathcal{A}}\big(f(B)\big). \tag{4}$$

Using Stone-Weierstrass Theorem, and the norm continuity of $q_{\mathcal{A}}$, it suffices to prove (4) in the case when f is a polynomial function. (Indeed, if we consider the compact set $\Omega = \mathrm{Spec}(A) \cup \mathrm{Spec}(B)$, then $f(A)$ and $f(B)$ depend only on the restriction $f\big|_{\Omega}$, and if we choose a sequence $(p_n)_{n=0}^{\infty}$ of polynomials in one variable, such that $f\big|_{\Omega} = \lim_{n\to\infty} p_n$ in $C(\Omega)$, then $f(A) = \lim_{n\to\infty} p_n(A)$ and $f(B) = \lim_{n\to\infty} p_n(B)$, in norm. Using the norm continuity of $q_{\mathcal{A}}$ we have $q_{\mathcal{A}}\big(f(A)\big) = \lim_{n\to\infty} q_{\mathcal{A}}\big(p_n(A)\big)$ and $q_{\mathcal{A}}\big(f(B)\big) = \lim_{n\to\infty} q_{\mathcal{A}}\big(p_n(B)\big)$.) When f is a polynomial function however, the equality (4) follows immediately from (vi) combined with the linearity of $q_{\mathcal{A}}$ on each of the abelian C*-subalgebras $C^*(\{I, A\})$ and $C^*(\{I, B\})$.

(v)⇒ (ii). Condition (v) implies

$$\int_{\mathbb{R}} f\, d\mu^A = \int_{\mathbb{R}} f\, d\mu^B,\ \ \forall\, f \in C_0(\mathbb{R}),$$

so it will clearly force $\mu^A = \mu^B$.

(ii)⇒ (iii) and (ii)⇒ (iv) are trivial because the conditions (iii) and (iv) read:

(iii) $\mu^A\big((-\infty, \lambda)\big) = \mu^B\big((-\infty, \lambda)\big)$, $\forall\, \lambda \in \mathbb{R}$;
(iv) $\mu^A\big((-\infty, \lambda]\big) = \mu^B\big((-\infty, \lambda]\big)$, $\forall\, \lambda \in \mathbb{R}$.

The same argument shows that we also have the implications (iii)⇒ (ii) and (iv)⇒ (ii), the reason being the fact that if one considers the collections

$$\mathcal{J}_1 = \big\{(-\infty, \lambda) \,:\, \lambda \in \mathbb{R}\big\},$$
$$\mathcal{J}_2 = \big\{(-\infty, \lambda] \,:\, \lambda \in \mathbb{R}\big\},$$

then for $k = 1, 2$ one has:

(*a*) $Bor(\mathbb{R}) = \mathbf{S}(\mathcal{J}_k)$ – the σ-ring generated by $\mathcal{J}_k$,
(*b*) $J, K \in \mathcal{J}_k \Rightarrow J \cap K \in \mathcal{J}_k$,

and then by standard arguments one has the implication

$$\mu^A\big|_{\mathcal{J}_k} = \mu^B\big|_{\mathcal{J}_k} \Rightarrow \mu^A = \mu^B.$$

(It is key here that both μ^A and μ^B are probability measures.)

(iii)⇒ (i). Assume condition (iii). Replacing A with $\delta A + \lambda I$, and B with $\delta B + \lambda I$, with $\delta, \lambda \in \mathbb{R} \smallsetminus \{0\}$ suitably chosen (use also Proposition 2.1), we can assume that $0 \le A, B \le \alpha I$ for some $\alpha \in (0, 1)$, that is,

$$\mathrm{Spec}(A) \cup \mathrm{Spec}(B) \subset [0, 1). \tag{5}$$

For every integer $n \ge 1$, consider then the spectral projections

$$P_{kn} = e_{[\frac{k-1}{n}, \frac{k}{n})}(A) \text{ and } Q_{kn} = e_{[\frac{k-1}{n}, \frac{k}{n})}(B),\ \ k = 1, \cdots, n,$$

which have dimensions

$$D(P_{kn}) = D\big(e_{(-\infty,\frac{k}{n})}(A)\big) - D\big(e_{(-\infty,\frac{k-1}{n})}(A)\big),$$
$$D(Q_{kn}) = D\big(e_{(-\infty,\frac{k}{n})}(B)\big) - D\big(e_{(-\infty,\frac{k-1}{n})}(B)\big).$$

Using the hypothesis (iii) we get $D(P_{kn}) = D(Q_{kn})$, so there exist partial isometries $V_{kn} \in \mathcal{A}$ with $V_{kn}V_{kn}^* = P_{kn}$ and $V_{kn}^*V_{kn} = Q_{kn}$. By (5) we also have the equalities

$$P_{1n} + P_{2n} + \cdots + P_{nn} = Q_{1n} + Q_{2n} + \cdots + Q_{nn} = I, \ \ \forall\, n \geq 2,$$

and then the element $U_n = \sum_{k=1}^n V_{kn}^*$ will be a unitary, satisfying

$$U_n P_{kn} U_n^* = Q_{kn}, \ \ \forall\, n \geq k \geq 1. \tag{6}$$

Using (5), for every $n \geq 1$, one has the inequalities

$$\sum_{k=1}^n \tfrac{k-1}{n} P_{kn} \leq A \leq \sum_{k=1}^n \tfrac{k}{n} P_{kn},$$
$$\sum_{k=1}^n \tfrac{k-1}{n} Q_{kn} \leq B \leq \sum_{k=1}^n \tfrac{k}{n} Q_{kn}.$$

In particular, the elements $A_n = \sum_{k=1}^n \frac{k}{n} P_{kn}$ and $B_n = \sum_{k=1}^n \frac{k}{n} Q_{kn}$, will satisfy

$$\|A_n - A\| \leq \tfrac{1}{n} \text{ and } \|B_n - B\| \leq \tfrac{1}{n}, \ \ \forall\, n \geq 1, \tag{7}$$

as well as:

$$U_n A_n U_n^* = B_n, \ \ \forall\, n \geq 1.$$

Using (7) we have

$$\|U_n A U_n^* - B\| \leq \|U_n A U_n^* - U_n A_n U_n^*\| + \|B_n - B\| \leq \tfrac{2}{n}, \ \ \forall\, n \geq 1,$$

so A and B are indeed approximatively unitarily equivalent. □

Corollary 2.1. *Let $\mathcal{A}$ be a finiteAW*-factor,and let $A_1, A_2, B_1, B_2 \in \mathcal{A}_{sa}$ be elements, with $A_1 \sim A_2$, $A_1 \perp B_1$, and $A_2 \perp B_2$. The following are equivalent:*

(i) $B_1 \sim B_2$;
(ii) $A_1 + B_1 \sim A_2 + B_2$.

Proof. Denote for simplicity $A_1 + B_1$ by X_1 and $A_2 + B_2$ by X_2. Using the orthogonality assumptions, one has the equalities $X_1^k = A_1^k + B_1^k$ and $X_2^k = A_2^k + B_2^k$, which in turn imply the equalities

$$q_{\mathcal{A}}(X_1^k) = q_{\mathcal{A}}(A_1^k) + q_{\mathcal{A}}(B_1^k) \text{ and } q_{\mathcal{A}}(X_2^k) = q_{\mathcal{A}}(A_2^k) + q_{\mathcal{A}}(B_2^k), \ \ \forall\, k \in \mathbb{N}.$$

Since we have $q_{\mathcal{A}}(A_1^k) = q_{\mathcal{A}}(A_2^k)$, it follows that the conditions

(i′) $q_{\mathcal{A}}(B_1^k) = q_{\mathcal{A}}(B_2^k)$, $\forall\, k \in \mathbb{N}$,
(ii′) $q_{\mathcal{A}}(X_1^k) = q_{\mathcal{A}}(X_2^k)$, $\forall\, k \in \mathbb{N}$,

are equivalent. By Theorem 2.1 however we have the equivalences (i) ⇔ (i′) and (ii) ⇔ (ii′). □

The following result is a slight (but useful) improvement of part (ii) from Proposition 2.1.

Proposition 2.3. *Let $\mathcal{A}$ be a finite AW*-factor.Assume $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ are nets in $\mathcal{A}_{sa}$, indexed by the same directed set Λ. Assume that:*

- *each of the nets $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ is bounded, abelian, and monotone;*
- $A_\lambda \sim B_\lambda$, $\forall\, \lambda \in \Lambda$.

If we consider the self-adjoint elements $A = \mathsf{W}\text{-}\lim_{\lambda\in\Lambda} A_\lambda$ *and* $B = \mathsf{W}\text{-}\lim_{\lambda\in\Lambda} B_\lambda$ *(which exist by Lemma 1.1), then* $A \sim B$.

Proof. Using Lemma 1.1, one has the equalities

$$q_{\mathcal{A}}(A^k) = \lim_{\lambda\in\Lambda} q_{\mathcal{A}}(A^k_\lambda) \text{ and } q_{\mathcal{A}}(B^k) = \lim_{\lambda\in\Lambda} q_{\mathcal{A}}(B^k_\lambda),\ \ \forall\, k \in \mathbb{N}.$$

Using the hypothesis $A_\lambda \sim B_\lambda$, and Theorem 2.1, we know that we have $q_{\mathcal{A}}(A^k_\lambda) = q_{\mathcal{A}}(B^k_\lambda)$, $\forall\, \lambda \in \Lambda$, $k \in \mathbb{N}$, so the above equalities give

$$q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(B^k),\ \ \forall\, k \in \mathbb{N},$$

and the desired conclusion follows again from Theorem 2.1. □

In preparation for the next definition, we introduce the following.

Notations. Let $\mathcal{A}$ be a C*-algebra. For $A \in \mathcal{A}_{sa}$, we denote by A^+ and A^- the positive and negative parts of A respectively. Recall that $A^\pm \in \mathcal{A}$ are two uniquely determined positive elements in $\mathcal{A}$, such that $A = A^+ - A^-$ and $A^+ \perp A^-$. (In fact $A^\pm = f^\pm(A)$, where $f^\pm : \mathbb{R} \to [0, \infty)$ are the continuous functions defined by $f^+(t) = \max\{t, 0\}$ and $f^-(t) = \max\{-t, 0\}$, $\forall\, t \in \mathbb{R}$.)

We now introduce the main concept used in this paper.

Definition. Let $\mathcal{A}$ be a unital C*-algebra.An element $A \in \mathcal{A}_{sa}$ is said to be *spectrally symmetric in* $\mathcal{A}$, if its positive and negative parts A^+ and A^- are approximately unitarily equivalent.

The following result, along the same lines as Theorem 2.1, gives several characterizations of spectral symmetry.

Theorem 2.2. *Let* $\mathcal{A}$ *be a finite AW*-factor.For an element* $A \in \mathcal{A}$*, the following are equivalent:*

(i) A *is spectrally symmetric;*
(ii) $A \sim -A$*;*
(iii) *the map* $\phi : \mathbb{R} \ni t \longmapsto -t \in \mathbb{R}$ *leaves the scalar spectral measure* μ^A *invariant, that is,* $\mu^A\big(\phi(B)\big) = \mu^A(B)$, $\forall\, B \in Bor(\mathbb{R})$*;*
(iv) $q_{\mathcal{A}}(A^k) = 0$*, for every odd non-negative integer* k.
(v) *there exist* $A_1, A_2 \in \mathcal{A}_{sa}$*, with* $A_1 \perp A_2$*,* $A_1 \sim A_2$*, and* $A_1 - A_2 = A$.

Proof. (i)⇒(v). This implication is trivial, by taking $A_1 = A^+$ and $A_2 = A^-$.

(v)⇒(iv). Assume $A = A_1 - A_2$, with A_1, A_2 as in (v). Since $A_1 \perp A_2$, it is pretty obvious that

$$A^k = A_1^k + (-1)^k A_2^k,\ \ \forall\, k \in \mathbb{N}.$$

Since by Theorem 2.1 we also have:

$$q_{\mathcal{A}}(A_1^k) = q_{\mathcal{A}}(A_2^k),\ \ \forall\, k \in \mathbb{N},$$

using the linearity of restriction of the quasitrace $q_{\mathcal{A}}$ to the abelian C*-subalgebra $C^*\{A_1, A_2\}$, we get

$$q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(A_1^k) - q_{\mathcal{A}}(A_2^k) = 0,$$

for every odd non-negative integer k.

(iv)⇒ (ii). Assume condition (iv), and let us prove that $A \sim -A$. Using Theorem 2.1, it suffices to show that $q_{\mathcal{A}}\big(A^k\big) = q_{\mathcal{A}}\big((-A)^k\big)$, or equivalently:

$$q_{\mathcal{A}}\big(A^k\big) = (-1)^k q_{\mathcal{A}}\big(A^k\big),\ \ \forall\, k \in \mathbb{N}.$$

For even k, this is trivial, while for odd k, this follows from (iv).

(ii)$\Leftrightarrow$ (iii). This equivalence is trivial, since the measure $\nu : Bor(\mathbb{R}) \to [0,1]$ defined by $\nu(B) = \mu^A\big(\phi(B)\big)$, $\forall\, B \in Bor(\mathbb{R})$, concides with the scalar spectral measure μ^{-A} of the self-adjoint element $-A$. Then condition (ii) is equivalent, by Theorem 2.1, to the equality $\mu^A = \nu$, which is precisely condition (iii).

(ii)$\Rightarrow$ (i). Assume $A \sim -A$, and let us prove that $A^+ \sim A^-$. If we consider the continuous function $f^+ : \mathbb{R} \ni t \longmapsto \max\{t, 0\} \in [0, \infty)$, then by Proposition 2.1, we know that $A^+ = f^+(A) \sim f^+(-A)$. The desired conclusion then follows from the obvious equality $f^+(-A) = A^-$. $\square$

Remark 2.1. If $A, B \in \mathcal{A}_{sa}$ are spectrally symmetric, and $A \perp B$, then $A + B$ is spectrally symmetric.

3. Scales of Projections and Riemann Integration

Definition. Let $\mathcal{A}$ be an AW*-factor of type II_1. A *scale of projections* in $\mathcal{A}$ is a system $\mathcal{E} = (E, J)$ consisting of a sub-interval $J \subset [0,1]$ and a map $E : J \to \mathbf{P}(\mathcal{A})$, with the following properties:

(i) $D\big(E(t)\big) = t$, $\forall\, t \in J$;

(ii) $t < s \Rightarrow E(t) \leq E(s)$.

Depending on the various features of the interval J, we say that

- the scale $\mathcal{E}$ is *closed*, if J is a closed interval;
- the scale $\mathcal{E}$ is *full*, if $J = [0,1]$.

Occasionally, we are going to abuse the notation and denote the collection of projections $\{E(t) \,:\, t \in J\}$ also by $\mathcal{E}$. (To avoid confusion, when we use this notation, we are going to use the phrase "$\mathcal{E}$ as a set.") For instance, given a projection $P \in \mathbf{P}(\mathcal{A})$ we are going to use the notation $P \in \mathcal{E}$ to indicate that $P = E(t)$, for some $t \in J$. Likewise, if $\mathcal{P} \subset \mathbf{P}(\mathcal{A})$ is a collection of projections, we use the notation $\mathcal{P} \subset \mathcal{E}$ to indicate that $P \in \mathcal{E}$, $\forall\, P \in \mathcal{P}$.

Remark 3.1. Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $J \subset [0,1]$ be a sub-interval. If $\mathcal{E} = (E, J)$ is a scale over J, then as a set $\mathcal{E}$ is totally ordered, and the map $E : J \to \mathcal{E}$ is bijective, so J is equal to the set $D(\mathcal{E}) = \big\{D(E) \,:\, E \in \mathcal{E}\big\}$. For this reason, the interval J will be referred to as the *dimension range of* $\mathcal{E}$.

Conversely, a totally ordered set of projections $\mathcal{E}$ is a scale if and only if the dimension range $D(\mathcal{E})$ is a sub-interval of $[0,1]$.

If $\mathcal{E} = (E, J)$ is a scale of projections, and if $J_0 \subset J$ is a sub-interval, the restriction $(E\big|_{J_0}, J_0)$ is clearly a scale, which will be denoted by $\mathcal{E}\big|_{J_0}$.

Remark 3.2. If $\mathcal{A}$ is an AW*-factor of type II_1, and $\mathcal{E}$ is a scale of projections in $\mathcal{A}$, with dimension range J, then there exists a unique closed scale $\overline{\mathcal{E}}$ with dimension $\overline{J}$ – the closure of J – with $\overline{\mathcal{E}}\big|_J = \mathcal{E}$. In fact, if $\overline{J} = [a, b]$, then the values at the endpoints, which by an abuse of notation will be denoted by $E(a)$ and $E(b)$, are given by $E(a) = \mathsf{W}\text{-}\lim_{t\to a^+} E(t)$ and $E(b) = \mathsf{W}\text{-}\lim_{t\to b^-} E(t)$. Because of this fact, for the remainder of this article we are going to deal exclusively with closed scales. The projections $E(a)$ and $E(b)$ will be referred to as the *initial* and *terminal* projections of the scale $\mathcal{E}$.

In preparation for subsequent constructions, we introduce the following:

Definitions. Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $\mathcal{E} = (E, J)$ be a scale of projections in $\mathcal{A}$.
A. Assuming the initial and terminal projections of $\mathcal{E}$ are F and G respectivley, we define the *width of* $\mathcal{E}$ to be the projection $\mathbf{w}(\mathcal{E}) = G - F$. The dimension of the width, that is, the number $D(\mathbf{w}(\mathcal{E}))$ – which is equal to the length of the dimension range – will be referred to as the *measure of* $\mathcal{E}$, and will be denoted by $m(\mathcal{E})$.
B. Given a projection $P \in \mathcal{A}$ with

$$P \leq E(t), \ \ \forall\, t \in J,$$

one can define the scale $\mathcal{E} - P = (F, K)$, where $K = J - D(P) = \big\{t - D(P) \,:\, t \in J\big\}$, and $F(t) = E(D(P) + t) - P$, $\forall\, t \in K$. The scale $\mathcal{E} - P$ is called the *downward translation of* $\mathcal{E}$ *by* P.
C. If $Q \in \mathcal{A}$ is a projection with $Q \perp E(t)$, $\forall\, t \in J$, one can define the scale $\mathcal{E} + Q = (G, L)$, where $L = J + D(Q) = \big\{t + D(Q) \,:\, t \in J\big\}$, and $G(t) = E(t - D(Q)) + Q$, $\forall\, t \in L$. The scale $\mathcal{E} + Q$ is called the *upward translation of* $\mathcal{E}$ *by* Q.

Comment. Let $\mathcal{A}$ be an AW*-factor of type II_1. Given a closed scale of projections $\mathcal{E}$ in $\mathcal{A}$, if we translate it downward by its initial projection, we obtain a new scale, denoted by $\tilde{\mathcal{E}}$, which has the same width, but which has initial projection 0. A scale with this property is said to be *normalized.* Most of our subsequent constructions will in effect depend only on the normalized scale.

Scales (as sets) are characterized as follows:

Proposition 3.1. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, and let $J \subset [0, 1]$ be a sub-interval. Consider the collection:*

$$\mathfrak{T}_{\mathcal{A}}(J) = \big\{\mathcal{P} \subset \mathbf{P}(\mathcal{A}) \,:\, \mathcal{P} \text{ totally ordered, and } D(P) \in J, \ \ \forall\, P \in \mathcal{P}\big\},$$

equipped with the inclusion order. For an element $\mathcal{E} \in \mathfrak{T}_{\mathcal{A}}(J)$, the following are equivalent:

(i) *$\mathcal{E}$ is a scale over J;*
(ii) *$\mathcal{E}$ is a maximal element in $\mathfrak{T}_{\mathcal{A}}(J)$.*

Proof. (i)$\Rightarrow$ (ii). Assume $\mathcal{E} = \{E(t) \,:\, t \in J\}$ is a scale over J, and let us show that $\mathcal{E}$ is maximal in $\mathfrak{T}_{\mathcal{A}}(J)$. Start with some $\mathcal{P} \in \mathfrak{T}_{\mathcal{A}}(J)$, with $\mathcal{P} \supset \mathcal{E}$, and let us prove that this forces the equality $\mathcal{P} = \mathcal{E}$. If we start with a projection $P \in \mathcal{P}$, and if we put $t = D(P)$, then by the definition we have $D\big(E(t)\big) = D(P)$. Put $X = E(t) - P$, and observe that, since both $E(t)$ and P are in $\mathcal{P}$, which is totally ordered, it follows that either X or $-X$ is a projection. In either case, the equality $D\big(E(t)\big) = D(P)$ will force $D(\pm X) = 0$, so we must have $X = 0$, i.e. $P = E(t)$, so P indeed belongs to $\mathcal{E}$.

(ii)$\Rightarrow$ (i). Assume $\mathcal{E}$ is a maximal element in $\mathfrak{T}_{\mathcal{A}}(J)$, and let us prove that $\mathcal{E}$ is a scale over J. By Remark 3.1, all we have to prove is the equality $D(\mathcal{E}) = J$. By construction we already have $D(\mathcal{E}) \subset J$, so we only need to prove the other inclusion. We argue by the contradiction. Assume there is some $s \in J$, such that

$$D(E) \neq s, \ \ \forall\, E \in \mathcal{E}. \tag{8}$$

Consider the collections of projections

$$\mathcal{F} = \big\{E \in \mathcal{E} \,:\, D(E) < s\big\} \text{ and } \mathcal{G} = \big\{E \in \mathcal{E} \,:\, D(E) > s\big\},$$

and define the numbers

$$r = \sup\big\{D(P) \,:\, P \in \{0\} \cup \mathcal{F}\big\},$$
$$t = \inf\big\{D(P) \,:\, Q \in \{I\} \cup \mathcal{G}\big\}.$$

(We add 0 and I simply because one of $\mathcal{F}$ or $\mathcal{G}$ could be empty.)

Claim. There exist projections $P \in \{0\} \cup \mathcal{F}$ and $Q \in \{I\} \cup \mathcal{G}$, such that $D(P) = r$, $D(Q) = t$.

To prove the existence of P, we may assume $r > 0$. In particular, $\mathcal{F} \neq \varnothing$, and $r = \sup D(\mathcal{F})$. Note that this, combined with the inequality $r \leq s$, forces $r \in J$. If we consider the set $D(\mathcal{F}) \subset [0,1]$, equipped with its natural order, then it becomes a directed set. For every $\lambda \in D(\mathcal{F})$ we choose $F_\lambda \in \mathcal{F}$ with $D(F_\lambda) = \lambda$ (by total ordering of $\mathcal{F}$, the projection F_λ is unique), so that we get a monotone net $(F_\lambda)_{\lambda \in D(\mathcal{F})}$ of projections. Since we work with projections, this forces $(F_\lambda)_{\lambda \in D(\mathcal{F})}$ to be both abelian and bounded, so using Lemma 1.1, this net has a weak limit. If we put $P = \mathsf{W}\text{-}\lim_{\lambda \in D(\mathcal{F})} F_\lambda$, then it is obvious that P is a projection, and moreover one has the equality $D(P) = r$. Remark that, since $F \leq G$, $\forall\, F \in \mathcal{F}$, $G \in \mathcal{G}$ (by total ordering), one gets the inequalities

$$F \leq P \leq G, \ \ \forall\, F \in \mathcal{F}, G \in \mathcal{G},$$

so the collection $\mathcal{E} \cup \{P\}$ is again totally ordered. Notice however that, since $D(P) = r \in J$, by maximality this forces $P \in \mathcal{E}$. Since $D(P) \leq s$, the condition (8) forces $P \in \mathcal{F}$. The existence of Q is proven in the exact same way with the reverse order relation.

Having proven the above Claim, let us observe that, by the arguments employed in the proof, we also have the inequalities

$$F \leq P \leq Q \leq G, \ \ \forall\, F \in \mathcal{F}, \ G \in \mathcal{G}. \tag{9}$$

Since we assume (8), it follows that $r < s < t$. Choose then (use the properties of AW*-algebras of type II_1) a projection $H \leq Q - P$ with $D(H) = s - r$, and define the projection $R = P + H$, so that

$$D(R) = D(P) + D(H) = s.$$

Since we obviously have $P \leq R \leq Q$, by (9) we also get

$$F \leq R \leq G, \ \ \forall\, F \in \mathcal{F}, \ G \in \mathcal{G},$$

so the set $\mathcal{E} \cup \{R\}$ is totally ordered. Since $D(R) = s \in J$, the maximality of $\mathcal{E}$ forces $R \in \mathcal{E}$, thus contradicting (8). ☐

Corollary 3.1. *Let $\mathcal{A}$ be an AW*-factor of type II_1, let $J \subset [0,1]$ be a sub-interval, and let $\mathcal{P} \in \mathfrak{T}_{\mathcal{A}}(J)$. There exists at least one scale $\mathcal{E}$ over J, with $\mathcal{E} \supset \mathcal{P}$.*

Proof. Immediate from Zorn's Lemma, and the above characterization. ☐

Remark 3.3. Here is an interesting special case of Corollary 3.1. Given $\mathcal{A}$ an AW*-factor of type II_1, and a self-adjoint element $A \in \mathcal{A}$, let us consider the collection

$$\mathfrak{S}(A) = \big\{e_{(-\infty,\alpha)}(A) \,:\, \alpha \in \mathbb{R}\big\} \cup \big\{e_{(-\infty,\beta]}(A) \,:\, \beta \in \mathbb{R}\big\}.$$

It is obvious that $\mathfrak{S}(A)$ is totally ordered. More precisely, one has the inequalities

$$e_{(-\infty,\beta)}(A) \leq e_{(-\infty,\beta]}(A) \leq e_{(-\infty,\alpha)}(A), \ \ \forall\, \alpha > \beta.$$

Notice that $\mathfrak{S}(A) \ni 0, I$, so by Corollary 3.1, there exists at least one full scale $\mathcal{E} \supset \mathfrak{S}(A)$. Such a (full) scale will be referred to as a *spectral scale for A* (in $\mathcal{A}$).

In preparation for the next construction, we introduce the following

Notations. Assume $\mathcal{A}$ is an AW*-factor of type II_1, and $\mathcal{E}$ is a closed scale of projections in $\mathcal{A}$, with dimension range $D(\mathcal{E}) = [a, b]$.

Given a partition

$$\Delta = [a = t_0 < t_1 < \cdots < t_n = b]$$

of the interval $[a, b]$, and a bounded function $f : [a, b] \to \mathbb{R}$, we define the lower and upper Darboux sums

$$L_{\mathcal{E}}(f, \Delta) = \sum_{k=1}^{n} \big[\inf_{s \in [t_{k-1}, t_k]} f(s) \big] \cdot \big[E(t_k) - E(t_{k-1}) \big],$$

$$U_{\mathcal{E}}(f, \Delta) = \sum_{k=1}^{n} \big[\sup_{s \in [t_{k-1}, t_k]} f(s) \big] \cdot \big[E(t_k) - E(t_{k-1}) \big].$$

Note that, for any partion Δ, using the linearity of $q_{\mathcal{A}}$ on the bicommutant $\mathcal{E}''$, one has the equalities

$$q_{\mathcal{A}}\big(L_{\mathcal{E}}(f, \Delta) = \sum_{k=1}^{n} \big[\inf_{s \in [t_{k-1}, t_k]} f(s) \big] \cdot [t_k - t_{k-1}] = L(f, \Delta), \tag{10}$$

$$q_{\mathcal{A}}\big(U_{\mathcal{E}}(f, \Delta) = \sum_{k=1}^{n} \big[\sup_{s \in [t_{k-1}, t_k]} f(s) \big] \cdot [t_k - t_{k-1}] = U(f, \Delta), , \tag{11}$$

where $L(f, \Delta)$ and $U(f, \Delta)$ denote the usual (scalar) lower and upper Darboux sums of f.

Observe also that, if we consider the set $\mathfrak{P}[a, b]$ of all partitions of $[a, b]$, ordered with respect to the inclusion, then $\mathfrak{P}[a, b]$ becomes a directed set, and moreover

- $\big(L_{\mathcal{E}}(f, \Delta)\big)_{\Delta \in \mathfrak{P}[a,b]}$ is an abelian increasing net,
- $\big(U_{\mathcal{E}}(f, \Delta)\big)_{\Delta \in \mathfrak{P}[a,b]}$ is an abelian decreasing net.

Since we also have the inequalities

$$\big[\inf_{s \in [a,b]} f(s) \big] \cdot \mathbf{w}(\mathcal{E}) \leq L_{\mathcal{E}}(f, \Delta) \leq U_{\mathcal{E}}(f, \Delta) \leq \big[\sup_{s \in [a,b]} f(s) \big] \cdot \mathbf{w}(\mathcal{E}), \ \forall \, \Delta \in \mathfrak{P}[a, b],$$

by Lemma 1.1 these nets are weakly convergent.

Proposition 3.2. *With the notations above, if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then one has the equalities*

$$\mathsf{W}\text{-}\lim_{\Delta \in \mathfrak{P}[a,b]} L_{\mathcal{E}}(f, \Delta) = \mathsf{W}\text{-}\lim_{\Delta \in \mathfrak{P}[a,b]} U_{\mathcal{E}}(f, \Delta).$$

Moreover, if we denote this common limit by A, then one has the equality

$$q_{\mathcal{A}}(A) = \int_a^b f(t)\, dt.$$

Proof. Put $L = \mathsf{W}\text{-}\lim_{\Delta \in \mathfrak{P}[a,b]} L_{\mathcal{E}}(f, \Delta)$ and $U = \mathsf{W}\text{-}\lim_{\Delta \in \mathfrak{P}[a,b]} U_{\mathcal{E}}(f, \Delta)$. If we consider the abelian von Neumann algebra $\mathcal{M} = \mathcal{E}''$, we have the equalities

$$L = \mathrm{w}^*_{\mathcal{M}} \lim_{\Delta \in \mathfrak{P}[a,b]} L_{\mathcal{E}}(f, \Delta) \text{ and } U = \mathrm{w}^*_{\mathcal{M}} \lim_{\Delta \in \mathfrak{P}[a,b]} U_{\mathcal{E}}(f, \Delta) \text{ (in } \mathcal{M}).$$

Using monotonicity of the nets $\big(L_{\mathcal{E}}(f,\Delta)\big)_{\Delta\in\mathfrak{P}[a,b]}$ and $\big(U_{\mathcal{E}}(f,\Delta)\big)_{\Delta\in\mathfrak{P}[a,b]}$, we also get the inequalities

$$L_{\mathcal{E}}(f,\Delta)\leq L\leq U\leq U_{\mathcal{E}}(f,\Delta),\ \ \forall\,\Delta\in\mathfrak{P}[a,b].$$

Using the order properties of the quasitrace (which is linear on $\mathcal{M}$), we have

$$q_{\mathcal{A}}\big(L_{\mathcal{E}}(f,\Delta)\big)\leq q_{\mathcal{A}}(L)\leq q_{\mathcal{A}}(U)\leq q_{\mathcal{A}}\big(U_{\mathcal{E}}(f,\Delta)\big),$$

which using (10) and (11) reads:

$$L(f,\Delta)\leq q_{\mathcal{A}}(L)\leq q_{\mathcal{A}}(U)\leq U(f,\Delta),\ \ \forall\,\Delta,\in\mathfrak{P}[a,b].$$

Taking limit this gives

$$q_{\mathcal{A}}(L)=q_{\mathcal{A}}(U)=\int_a^b f(t)\,dt.$$

In particular (use the linearity of the quasitrace on $\mathcal{M}$), this gives $q_{\mathcal{A}}(U-L)=0$, and then the inequality $U-L\geq 0$, combined with the faitfulness of the quasitrace, will force $U=L$. ☐

Notation. Given a closed scale $\mathcal{E}$ as above – with dimension range $[a,b]$ – and a Riemann integrable function $f:[a,b]\to\mathbb{R}$, the element $A\in\mathcal{E}''$, defined in the above result, will be denoted by $\int_a^b f(t)\,dE(t)$ (or simply $\int_a^b f\,d\mathcal{E}$, when there is no danger of confusion). If we denote by $\mathfrak{R}[a,b]$ the algebra of real-valued Riemann integrable functions, the correspondence

$$\mathfrak{R}[a,b]\ni f\longmapsto\int_a^b f\,d\mathcal{E}\tag{12}$$

will be referred to as the *Riemann integral calculus associated with the scale $\mathcal{E}$.*

Remark 3.4. Given a scale $\mathcal{E}$ with dimension range $[a,b]$, and $f\in\mathfrak{R}[a,b]$, the element $A=\int_a^b f\,d\mathcal{E}$ will satisfy the inequalities

$$\big[\inf_{s\in[a,b]}f(s)\big]\cdot\mathbf{w}(\mathcal{E})\leq A\leq\big[\inf_{s\in[a,b]}f(s)\big]\cdot\mathbf{w}(\mathcal{E}).$$

(This follows from the corresponding inequalities for lower and upper Darboux sums, after taking weak limit in $\mathcal{E}''$.) This will then give the inequality

$$\mathbf{s}(A)\leq\mathbf{w}(\mathcal{E}),$$

where $\mathbf{s}(A)$ denotes the *support of* A. (Recall that, given an AW*-algebra $\mathcal{A}$ and an element $A\in\mathcal{A}_{sa}$, one defines $\mathbf{s}(A)=I-P$, where $P\in\mathbf{P}(\mathcal{A})$ is the projection defined by the condition $\mathbf{L}\big(\{A\}\big)=\mathcal{A}P$. Equivalently, using Borel functional calculus, $\mathbf{s}(A)=e_{\mathbb{R}\smallsetminus\{0\}}(A)$.)

The following technical result deals with sequential approximation.

Lemma 3.1. *Let $\mathcal{A}$ be an AW*-factor of type II_1, let $\mathcal{E}$ be a scale of projections in $\mathcal{A}$ with dimension range $[a,b]$, and let $f\in\mathfrak{R}[a,b]$. Assume $(\Delta_n)_{n=1}^\infty$ is a sequence of partitions of $[a,b]$, with $\Delta_1\subset\Delta_2\subset\dots$.*

(i) *If $\int_a^b f(t)\,dt=\lim_{n\to\infty}L(f,\Delta_n)$, then $\int_a^b f\,d\mathcal{E}=\mathsf{W}\text{-}\lim_{n\to\infty}L_{\mathcal{E}}(f,\Delta_n)$.*
(ii) *If $\int_a^b f(t)\,dt=\lim_{n\to\infty}U(f,\Delta_n)$, then $\int_a^b f\,d\mathcal{E}=\mathsf{W}\text{-}\lim_{n\to\infty}U_{\mathcal{E}}(f,\Delta_n)$.*

Proof. It suffices to prove property (i). (To prove (ii) we simply use (i) with f replaced by $-f$.)

To prove (i), denote $\int_a^b f\,d\mathcal{E}$ simply by B, and define the sequence $(A_n)_{n=1}^\infty \subset \mathcal{E}''$ by $A_n = L_{\mathcal{E}}(f,\Delta_n)$, $n \in \mathbb{N}$. On the one hand, since the sequence $(\Delta_n)_{n=1}^\infty$ is increasing, the sequence $(A_n)_{n=1}^\infty$ is increasing. On the other hand, it is clear that we have the inequalities

$$A_n \le B,\ \ \forall\, n \in \mathbb{N}. \tag{13}$$

Using Lemma 1.1, the limit $A = \mathsf{W}\text{-}\lim_{n\to\infty} A_n$ exists, and it will have quasitrace

$$q_{\mathcal{A}}(A) = \lim_{n\to\infty} q_{\mathcal{A}}(A_n) = \lim_{n\to\infty} L(f,\Delta_n) = \int_a^b f(t)\,dt = q_{\mathcal{A}}(B). \tag{14}$$

Finally, working in $\mathcal{E}''$, (13) yields $A \le B$. By the linearity and faithfulness of $q_{\mathcal{A}}$ on $\mathcal{E}''$, the equality (14) will force $A = B$. $\square$

Remark 3.5. The construction of the element $A = \int_a^b f\,d\mathcal{E}$ is compatible with translations. To be more precise, if one defines the translation maps $\Lambda_s : \mathfrak{R}[a,b] \to \mathfrak{R}[a+s,b+s]$ by

$$(\Lambda_s f)(t) = f(t-s),\ \ \forall\, t \in [a-s,b-s],\ f \in \mathfrak{R}[a,b],$$

then one has the following properties:

(i) If $\mathcal{E} = (E,[a,b])$ is a scale, and P is a projection with $P \le E(a)$, with dimension $D(P) = \delta$, then
$$\int_a^b f\,d\mathcal{E} = \int_{a-\delta}^{b-\delta} \Lambda_{-\delta} f\,d(\mathcal{E}-P),\ \ \forall\, f \in \mathfrak{R}[a,b].$$

(ii) If $\mathcal{E} = (E,[a,b])$ is a scale, and Q is a projection with $Q \perp E(b)$, with dimension $D(Q) = \delta$, then
$$\int_a^b f\,d\mathcal{E} = \int_{a+\delta}^{b+\delta} \Lambda_{\delta} f\,d(\mathcal{E}+Q),\ \ \forall\, f \in \mathfrak{R}[a,b].$$

Notation. Given a scale $\mathcal{E}$ with dimension range $[a,b]$, a function $f \in \mathfrak{R}[a,b]$, and a sub-interval $[a_1,b_1] \subset [a,b]$, we are going to denote by $\int_{a_1}^{b_1} f\,d\mathcal{E}$ the element $\int_a^b e_{[a_1,b_1]} f\,d\mathcal{E}$. An equivalent description can be given in terms of restriction:

$$\int_{a_1}^{b_1} \big(f\big|_{[a_1,b_1]}\big)\,d\big(\mathcal{E}\big|_{[a_1,b_1]}\big).$$

The following result summarizes several easy properties of this calculus.

Proposition 3.3. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, and let $\mathcal{E}$ be a closed scale of projections, with dimension range $D(\mathcal{E}) = [a,b]$.*

(i) *The map* (12) *is a real algebra homomorphism.*

(ii) *One has the inequality*
$$\Big\|\int_a^b f\,d\mathcal{E}\Big\| \le \|f\|_{\sup},\ \ \forall\, f \in \mathfrak{R}[a,b],$$
where $\|\,.\,\|_{\sup}$ stands for the supremum norm.

(iii) *Given a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, one has the equalities*

$$\phi\Big(\int_a^b f\, d\mathcal{E}\Big) = \int_a^b (\phi \circ f)\, d\mathcal{E}, \ \ \forall\, f \in \mathfrak{R}[a,b],$$

where the left hand side is obtained by continuos functional calculus applied to the self-adjoint element $\int_a^b f\, d\mathcal{E}$.

Proof. (i). To prove additivity, we start with two Riemann integrable functions $f_1, f_2 : [a,b] \to \mathbb{R}$, and we prove the equality

$$\int_a^b (f_1 + f_2)\, d\mathcal{E} = \int_a^b f_1\, d\mathcal{E} + \int_a^b f_2\, d\mathcal{E}. \tag{15}$$

If we work in the von Neumann algebra $\mathcal{M} = \langle\langle \mathcal{E} \rangle\rangle$, the for every partition $\Delta \in \mathfrak{P}[a,b]$, one obviously has the inequalities

$$L_{\mathcal{E}}(f_1,\Delta) + L_{\mathcal{E}}(f_2,\Delta) \leq L_{\mathcal{E}}(f_1+f_2,\Delta) \leq U_{\mathcal{E}}(f_1+f_2,\Delta) \leq U_{\mathcal{E}}(f_1,\Delta) + U_{\mathcal{E}}(f_2,\Delta),$$

so taking $\mathrm{w}^*_{\mathcal{M}}$-limit will give (15).

The homogeneity property

$$\int_a^b (\alpha f)\, d\mathcal{E} = \alpha \int_a^b f\, d\mathcal{E}, \ \ \forall\, \alpha \in \mathbb{R},\ f \in \mathfrak{R}[a,b]$$

is proven in the exact same way.

In order to prove that the correspondence (12) is multiplicative, it suffices to prove that it has the property:

$$\int_a^b f^k\, d\mathcal{E} = \Big[\int_a^b f\, d\mathcal{E}\Big]^k, \ \ \forall\, f \in \mathfrak{R}[a,b],\ k \in \mathbb{N}. \tag{16}$$

Using the obvious equality

$$\int_a^b 1\, d\mathcal{E} = \mathbf{w}(\mathcal{E}),$$

and the linearity, it may assume in (16) that $f \geq 0$. If we fix such an f, as well as $k \in \mathbb{N}$, and we define the net $(A_\Delta)_{\Delta \in \mathfrak{P}[a,b]}$ by

$$A_\Delta = L_{\mathcal{E}}(f,\Delta)^k,$$

then, on the one hand, by Lemma 1.1 we know that

$$\underset{\Delta \in \mathfrak{P}[a,b]}{\mathsf{W}\text{-}\lim} A_\Delta = \Big[\int_a^b f\, d\mathcal{E}\Big]^k.$$

On the other hand, using the fact that f is non-negative, it is quite clear that

$$A_\Delta = L_{\mathcal{E}}(f^k,\Delta), \ \ \forall\, \Delta \in \mathfrak{P}[a,b],$$

so we get

$$\underset{\Delta \in \mathfrak{P}[a,b]}{\mathsf{W}\text{-}\lim} A_\Delta = \int_a^b f^k\, d\mathcal{E}.$$

(ii). This inequality is trivial.

(iii). Fix $f \in \mathfrak{R}[a,b]$, as well as a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and denote $\int_a^b f\, \mathcal{E}$ simply by A. Using the Stone-Weierstrass Theorem, we know that for every $\varepsilon > 0$, there exists a polynomial function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$, such that

$$\big|\phi(s) - \phi_\varepsilon(s)\big| \leq \varepsilon, \ \ \forall\, s \in \big[-\|f\|_{\sup}, \|f\|_{\sup}\big].$$

On the one hand, by (ii) we have $\mathrm{Spec}(A) \subset \big[- \|f\|_{\sup}, \|f\|_{\sup}\big]$, so using the properties of functional calculus we know that

$$\big\|\phi(A) - \phi_\varepsilon(A)\big\| \leq \varepsilon. \tag{17}$$

On the other hand, using (i) we know that

$$\int_a^b (\phi_\varepsilon \circ f)\, d\mathcal{E} = \phi_\varepsilon(A). \tag{18}$$

Finally, since we obviously have

$$\big|(\phi \circ f)(t) - (\phi_\varepsilon \circ f)(t)\big| \leq \varepsilon, \ \ \forall\, t \in [a, b],$$

we also have the inequality

$$\Big\| \int_a^b (\phi \circ f)\, d\mathcal{E} - \int_a^b (\phi_\varepsilon \circ f)\, d\mathcal{E}\Big\| \leq \varepsilon,$$

so using (17) and (18) we get

$$\Big\|\phi(A) - \int_a^b (\phi \circ f)\, d\mathcal{E}\Big\| \leq 2\varepsilon.$$

Since this inequality is true for every $\varepsilon > 0$, it forces

$$\phi(A) = \int_a^b (\phi \circ f)\, d\mathcal{E}. \quad \square$$

Corollary 3.2. *With the notations above, if $\mathcal{E}$ is a scale with dimension range $[a, b]$, and if $f, g \in \mathfrak{R}[a, b]$ are such that*

$$f = g, \ \ \text{(Lebesgue) a.e.}$$

then $\int_a^b f\, d\mathcal{E} = \int_a^b g\, d\mathcal{E}$.

Proof. If one considers the commuting elements $X = \int_a^b f\, d\mathcal{E}$ and $Y = \int_a^b g\, d\mathcal{E}$, then the positive element $(X - Y)^2$ will have quasitrace

$$q_{\mathcal{A}}\big((X - Y)^2\big) = \int_a^b [f(t) - g(t)]^2\, dt = 0,$$

which obviousy forces $X = Y$. $\square$

Corollary 3.3. *Let $\mathcal{A}$ be an AW*-factor of type II_1, let $\mathcal{F}$ and $\mathcal{G}$ be closed scales of projections in $\mathcal{A}$ with dimension ranges $[a, b]$ and $[c, d]$ respectively, and let $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ be two Riemann integrable functions, such that*

$$\int_a^b f(t)^k\, dt = \int_c^d g(s)^k\, ds, \ \ \forall\, k \in \mathbb{N}. \tag{19}$$

Then the elements $A = \int_a^b f\, d\mathcal{F}$ and $B = \int_c^d g\, d\mathcal{G}$ are approximately unitary equivalent.

Proof. By the properties of the Riemann calculus, for every $k \in \mathbb{N}$, we have

$$A^k = \int_a^b f^k\, d\mathcal{F} \text{ and } B^k = \int_a^b g^k\, d\mathcal{G},$$

so using (19) we get

$$q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(B^k), \ \ \forall\, k \in \mathbb{N}.$$

From Theorem 2.1 it follows that $A \sim B$. $\square$

The next result should be regarded as a "Change of Variable" rule. In preparation for its formulation, we introduce the following terminology.

Notations. Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $P \in \mathbf{P}(\mathcal{A})$ be a non-zero projection. We define the compression

$$P\mathcal{A}P = \big\{PAP \,:\, A \in \mathcal{A}\big\}.$$

Of course, $P\mathcal{A}P$ is an AW*-subalgebra of $\mathcal{A}$, with unit P, but it is also a *factor*, so in fact $P\mathcal{A}P$ is itself an AW*-factor of type II_1. Its quasitrace is then given by

$$q_{P\mathcal{A}P}(X) = \frac{q_{\mathcal{A}}(X)}{D_{\mathcal{A}}(P)},\ \ \forall\, X \in P\mathcal{A}P, \tag{20}$$

Proposition 3.4. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, and let $P \in \mathbf{P}(\mathcal{A})$ be a non-zero projection. Put $\lambda = D(P)$.*

(i) *If $\mathcal{E} = (E, [a,b])$ is a scale of projections in $P\mathcal{A}P$, then the map $E^P : [\lambda a, \lambda b] \to \mathbf{P}(\mathcal{A})$, given by*

$$E^P(t) = E(t/\lambda),\ \ \forall\, t \in [\lambda a, \lambda b],$$

defines a scale $\mathcal{E}^P = \big(E^P, [\lambda a, \lambda b]\big)$ in $\mathcal{A}$. Moreover, for any $f \in \mathfrak{R}[\lambda a, \lambda b]$, one has the equality

$$\int_{\lambda a}^{\lambda b} f(t)\, dE^P(t)\, (\text{in } \mathcal{A}) \ = \int_a^b f(\lambda t)\, dE(t)\, (\text{in } P\mathcal{A}P). \tag{21}$$

(ii) *Conversely, if $\mathcal{F} = (F, [\alpha, \beta])$ is a scale in $\mathcal{A}$, with $F(\beta) \le P$, then the map $E : [\alpha/\lambda, \beta/\lambda] \to \mathbf{P}(P\mathcal{A}P)$, given by*

$$E(t) = F(\lambda t),\ \ \forall\, t \in [\alpha/\lambda, \beta/\lambda],$$

defines a scale $\mathcal{E} = \big(E, [\alpha/\lambda, \beta/\lambda]\big)$ in $P\mathcal{A}P$, with $\mathcal{E}^P = \mathcal{F}$.

Proof. (i). The fact that $\mathcal{E}^P$ is a scale in $\mathcal{A}$ is quite clear, since by (20) we have

$$D_{\mathcal{A}}\big(E^P(t)\big) = D_{\mathcal{A}}\big(E(t/\lambda)\big) = D_{\mathcal{A}}(P)\cdot D_{P\mathcal{A}P}\big(E(t/\lambda)\big) = \lambda\cdot(t/\lambda) = t,\ \ \forall\, t \in [\lambda a, \lambda b].$$

To prove the statement about Riemann integrals, we use the following notations:

- given a partition $\Delta \in \mathfrak{P}[a,b]$, say $\Delta = [a = t_0 < \cdots < t_n = b]$, we define the partition $\lambda\Delta \in \mathfrak{P}[\lambda a, \lambda b]$ as $\lambda\Delta = [\lambda a = \lambda t_0 < \cdots < \lambda t_n = \lambda b]$;
- for a function $f : [\lambda a, \lambda b] \to \mathbb{R}$ we define the function $f^\lambda : [a,b] \to \mathbb{R}$ by $f^\lambda(t) = f(\lambda t)$, $\forall\, t \in [a,b]$.

With these notations, one has the following easy facts:

(A) the correspondence $\mathfrak{P}[a,b] \ni \Delta \longmapsto \lambda\Delta \in \mathfrak{P}[\lambda a, \lambda b]$ is an order preserving bijection;

(B) for a bounded function $f : [\lambda a, \lambda b] \to \mathbb{R}$, the function $f^\lambda : [a,b] \to \mathbb{R}$ is bounded, and, for any sub-interval $[c,d] \subset [a,b]$, one has the equalities

$$\inf_{s \in [\lambda c, \lambda d]} f(s) = \inf_{t \in [c,d]} f^\lambda(t) \ \text{ and } \sup_{s \in [\lambda c, \lambda d]} f(s) = \sup_{t \in [c,d]} f^\lambda(t).$$

Using the above two facts we see that $f \in \mathfrak{R}[\lambda a, \lambda b] \Leftrightarrow f \in \mathfrak{R}[a,b]$, and moreover, one has the equalities

$$L_{\mathcal{E}^P}(f, \lambda\Delta) = L_{\mathcal{E}}(f^\lambda, \Delta) \text{ and } U_{\mathcal{E}^P}(f, \lambda\Delta) = U_{\mathcal{E}}(f^\lambda, \Delta),\ \ \forall\, \Delta \in \mathfrak{P}[a,b].$$

Taking weak limits in $\mathcal{A}$ and $P\mathcal{A}P$ respectively, then yields the equality

$$\int_{\lambda a}^{\lambda b} f \, d\mathcal{E}^P \text{ (in } \mathcal{A}) = \int_a^b f^\lambda \, d\mathcal{E} \text{ (in } P\mathcal{A}P),$$

which is precisely (21).

(ii). This statement is trivial. $\square$

The Riemann integral calculus developed above will be used in connection with the following key result.

Theorem 3.1. *Let $\mathcal{A}$ be an AW*-factor of type* II_1 *and let $A \in \mathcal{A}_{sa}$.*

(i) *The function $\omega_A : [0,1] \to \mathbb{R}$, defined by*

$$\omega_A(t) = \begin{cases} \min \operatorname{Spec}(A) & \text{if } t = 0 \\ \inf \left\{ \alpha \in \mathbb{R} \,:\, D\big(e_{(-\infty,\alpha]}(A)\big) \geq t \right\} & \text{if } t \in (0,1] \end{cases} \tag{22}$$

in non-decreasing, hence Riemann integrable. Moreover, one has the equality $\omega_A(1) = \max$ Spec(A).

(ii) *For any integer $k \geq 1$, one has the equality*

$$q_{\mathcal{A}}(A^k) = \int_0^1 \omega_A(t)^k \, dt.$$

(iii) *For any spectral scale $\mathcal{E}$ for A, one has the equality*

$$A = \int_0^1 \omega_A(t) \, d\mathcal{E}(t). \tag{23}$$

Proof. The fact that $\omega_A\big|_{(0,1]}$ is non-decreasing is trivial. The equality $\omega_A(1) = \max \operatorname{Spec}(A)$ is pretty obvious, since the inequality $D\big(e_{(-\infty,\alpha]}(A)\big) \geq 1$ is equivalent to the equality $e_{(-\infty,\alpha]} = I$, which in turn is equivalent to $A \leq \alpha I$. To finish the proof of (i), we fix some $t \in (0,1]$, and we must show that $\omega_A(0) \leq \omega_A(t)$. We argue by contradiction, assuming $\omega_A(0) > \omega_A(t)$, so there exists $\alpha \in \mathbb{R}$ with $\omega_A(0) > \alpha$, and $D\big(e_{(-\infty,\alpha]}(A)\big) \geq t (> 0)$. This is however impossible, since the inequality $\alpha < \omega_A(0)$ forces $e_{(-\infty,\alpha]}(A) \leq e_{(-\infty,\omega_A(0))}(A)$, and by construction $e_{(-\infty,\omega_A(0))}(A) = 0$.

Using Proposition 3.3, it is clear that property (ii) follows from property (iii).

To prove property (iii), we start off by fixing a spectral scale (see Remark 3.3) $\mathcal{E} = \big(E(t)\big)_{t \in [0,1]}$ for A.

Claim 1. For every $t \in [0,1]$ one has the inequalities

$$e_{(-\infty,\omega_A(t))}(A) \leq E(t) \leq e_{(-\infty,\omega_A(t)]}(A). \tag{24}$$

Since both projections $e_{(-\infty,\omega_A(t))}(A)$ and $e_{(-\infty,\omega_A(t)]}(A)$ belong to $\mathfrak{S}(A) \subset \mathcal{E}$, by total ordering, all we have to prove are the corresponding inequalities for the dimensions, i.e.

$$D\big(e_{(-\infty,\omega_A(t))}(A)\big) \leq t \leq D\big(e_{(-\infty,\omega_A(t)]}(A)\big),$$

or equivalently, using the scalar spectral measure,

$$\mu^A\big((-\infty,\omega_A(t))\big) \leq t \leq \mu^A\big((-\infty,\omega_A(t)]\big). \tag{25}$$

Since $\omega_A(0) = \min \operatorname{Spec}(A)$, we have $\mu^A\big((-\infty, \omega_A(0))\big) = 0$, so (25) is trivial for $t = 0$. Assume now $t \in (0, 1]$. To prove the inequalities (25) we consider the non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(\alpha) = \mu^A\big((-\infty, \alpha)\big) \text{ and } g(\alpha) = \mu^A\big((-\infty, \alpha]\big), \ \forall\, \alpha \in \mathbb{R},$$

and we consider the set $\Omega_t = \big\{\alpha \in \mathbb{R} : g(\alpha) \geq t\big\}$, so that $\omega_A(t) = \inf \Omega_t$. Since μ^A is a measure on $Bor(\mathbb{R})$, we know that g is continuous from the right, i.e.

$$g(\beta) = \lim_{\alpha \to \beta^+} g(\alpha), \ \forall\, \beta \in \mathbb{R}.$$

In particular, we have $g\big(\omega_A(t)\big) = \lim_{\alpha \to \omega_A(t)^+} g(\alpha) \geq t$, which gives the second inequality in (25). Since we also have

$$f(\beta) = \lim_{\alpha \to \beta^-} g(\alpha), \ \forall\, \beta \in \mathbb{R}.$$

and $g(\alpha) < t$, $\forall\, \alpha < \omega_A(t)$, we immediately get $f\big(\omega_A(t)\big) \leq t$, which is the first inequality in (25).

Claim 2. $A \in \mathcal{E}'$*, i.e.* $AE(t) = E(t)A$, $\forall\, t \in [0, 1]$.

Fix $t \in [0, 1]$, and notice that, since $e_{(-\infty, \omega_A(t))}(A)$ commutes with A, it suffices to show that $F(t) = E(t) - e_{(-\infty, \omega_A(t))}(A)$ commutes with A. By Claim 1 it follows that $F(t)$ is a projection, and moreover,

$$F(t) \leq e_{(-\infty, \omega_A(t)]}(A) - e_{(-\infty, \omega_A(t))}(A) = e_{\{\omega_A(t)\}}(A).$$

This obviously forces $F(t)A = \omega_A(t)F(t) = AF(t)$, and we are done.

Claim 3. For any partition $\Delta \in \mathfrak{P}[0, 1]$*, the lower and upper Darboux sums of* ω_A *satisfy the inequalities*

$$L_{\mathcal{E}}(\omega_A, \Delta) \leq A \leq U_{\mathcal{E}}(\omega_A, \Delta). \tag{26}$$

On the one hand, if $\Delta = (0 = t_0 < t_1 < \cdots < t_n = 1)$, due to the monotonicity of ω_A, one has the equalities

$$L_{\mathcal{E}}(\omega_A, \Delta) = \sum_{k=1}^{n} \omega_A(t_{k-1})\big[E(t_k) - E(t_{k-1})\big], \tag{27}$$

$$U_{\mathcal{E}}(\omega_A, \Delta) = \sum_{k=1}^{n} \omega_A(t_k)\big[E(t_k) - E(t_{k-1})\big], . \tag{28}$$

On the other hand, using Claim 1, we have the inequalities

$$e_{(-\infty, \omega_A(t_{k-1}))}(A) \leq E(t_{k-1}) \leq E(t_k) \leq e_{(-\infty, \omega_A(t_k)]}(A),$$

which gives the inequalities

$$E(t_k) - E(t_{k-1}) \leq e_{[\omega_A(t_{k-1}), \omega_A(t_k)]}(A), \ \forall\, k = 1, \dots, n. \tag{29}$$

Of course, the spectral projections $e_{[\omega_A(t_{k-1}), \omega_A(t_k)]}(A)$ satisfy the inequalities

$$\omega_A(t_{k-1})e_{[\omega_A(t_{k-1}), \omega_A(t_k)]}(A) \leq Ae_{[\omega_A(t_{k-1}), \omega_A(t_k)]}(A) \leq \omega_A(t_k)e_{[\omega_A(t_{k-1}), \omega_A(t_k)]}(A),$$

so multiplying this inequality by $E(t_k) - E(t_{k-1})$, which by Claim 2 commutes with all three sides, and using (29) we get

$$\omega_A(t_{k-1})\big[E(t_k) - E(t_{k-1})\big] \leq A\big[E(t_k) - E(t_{k-1})\big] \leq \omega_A(t_k)\big[E(t_k) - E(t_{k-1})\big],$$

for all $k = 1, \dots, n$. Summing up, using the obvious equality

$$\sum_{k=1}^{n} \big[E(t_k) - E(t_{k-1})\big] = E(1) - E(0) = I,$$

as well as (27) and (28), the desired inequalities (26) immediately follow.

After all these preparations, we proceed with the proof of (23). First of all, we notice that by Claim 2 we know that $\{A\} \cup \mathcal{E}$ is involutive and abelian, the AW*-subalgebra $\mathcal{M} = \big(\{A\} \cup \mathcal{E}\big)''$ is an abelian von Neumann algebra. Secondly, if we consider the element $B = \int_0^1 \omega_A(t)\, d\mathcal{E}(t)$, then A and B belong to $\mathcal{M}$. Moreover, since one has the equalities

$$B = \mathrm{w}^*_{\mathcal{M}}\text{-}\lim_{\Delta \in \mathfrak{P}[0,1]} L_{\mathcal{E}}(\omega_A, \Delta) = \mathrm{w}^*_{\mathcal{M}}\text{-}\lim_{\Delta \in \mathfrak{P}[0,1]} U_{\mathcal{E}}(\omega_A, \Delta),$$

by Claim 3 we must have both inequalities $B \leq A$ and $A \leq B$, so we indeed have the equality $A = B$. □

The result below – essentially a converse of Remark 3.4 – is useful when estimating the dimension of the support.

Proposition 3.5. *Given an AW*-factor $\mathcal{A}$ of type II_1, and a non-zero element $A \in \mathcal{A}_{sa}$, there exist*

(i) *a scale $\mathcal{E} = \big(E, [0, \delta]\big)$ with $E(\delta) = \mathbf{s}(A)$, and*
(ii) *a non-decreasing function $f : [0, \delta] \to \big[\min \mathrm{Spec}(A), \max \mathrm{Spec}(A)\big]$, such that $A = \int_0^\delta f\, d\mathcal{E}$.*

(Note that (ii) in fact forces $D\big(\mathbf{s}(A)\big) = \delta$.)

Proof. Denote $\mathbf{s}(A)$ simply by P, and let $D(P) = \delta$. Since $A \in P\mathcal{A}P$, one can write

$$A = \int_0^1 g\, d\mathcal{F}\, (\text{in } P\mathcal{A}P),$$

where $\mathcal{F} = (F, [0, 1])$ is a spectral scale for A in $P\mathcal{A}P$ (so $F(1) = P$), and $g \in \mathfrak{R}[0, 1]$ is some non-decreasing function, namely ω_A, but computed in $P\mathcal{A}P$. Of course, since $\mathrm{Spec}_{P\mathcal{A}P}(A) \subset \mathrm{Spec}(A)$, one has:

$$\min \mathrm{Spec}(A) \leq g(t) \leq \max \mathrm{Spec}(A), \ \ \forall\, t \in [0, 1]. \tag{30}$$

Using Proposition 3.4, if we consider $\mathcal{E} = \mathcal{F}^P$, namely $\mathcal{E} = (E, [0, \delta])$, with

$$E(t) = F(t/\delta), \ \ \forall\, t \in [0, \delta],$$

and if we define the function $f \in \mathfrak{R}[0, \delta]$ by

$$f(t) = g(t/\delta), \ \ \forall\, t \in [0, \delta],$$

then by Proposition 3.4 we get:

$$\int_0^\delta f(t)\, dE(t)\, (\text{in } \mathcal{A}) \,=\, \int_0^1 f(\delta t)\, dF(t)\, (\text{in } P\mathcal{A}P) \,=\, \int_0^1 g\, d\mathcal{F}\, (\text{in } P\mathcal{A}P) \,=\, A.$$

Using (30), we also have the inclusion $\mathrm{Range}\, f \subset \big[\min \mathrm{Spec}(A), \max \mathrm{Spec}(A)\big]$. Finally, the equality $E(\delta) = F(1) = P$ is trivial. □

We conclude this section with two applications of the Riemann calculus. One application (Proposition 3.6 below) deals with "copying" elements. The second one (Example 3.1) shows hwo to build self-adjoint elements with prescribed scalar spectral measure.

Before we discuss the next result, let us introduce the following terminology.

Definition. Let $\mathcal{A}$ be a finite AW*-factor. A subalgebra $\mathcal{B} \subset \mathcal{A}$ is said to be an *AW*-subfactor of* $\mathcal{A}$, if:

- $\mathcal{B}$ is an AW*-subalgebra of $\mathcal{A}$, which contains I – the unit of $\mathcal{A}$;
- $\mathcal{B}$ is a factor.

Comments. Let $\mathcal{A}$ be a finite AW*-factor.

A. If $\mathcal{B}$ is an AW*-subfactor of $\mathcal{A}$, then $\mathcal{B}$ is obviously a finite AW*-factor, and its canonical quasitrace (by uniqueness) is given by $q_{\mathcal{B}} = q_{\mathcal{A}}\big|_{\mathcal{B}}$ – the restriction of $q_{\mathcal{A}}$ to $\mathcal{B}$. In particular, in the case when $\mathcal{B}$ is of type II_1(this forces $\mathcal{A}$ to be of type II_1as well), for a collection $\mathcal{E} \subset \mathcal{B}$, the conditions
 (a) $\mathcal{E}$ is a scale of projections in $\mathcal{A}$, and
 (b) $\mathcal{E}$ is a scale of projections in $\mathcal{B}$,
 are equivalent. Moreover, if $\mathcal{E}$ has dimension range $[a, b]$, then
$$\int_a^b f\, d\mathcal{E}\,(\text{in } \mathcal{B}) \,=\, \int_a^b f\, d\mathcal{E}\,(\text{in } \mathcal{A}),\ \ \forall\, f \in \mathfrak{R}[a, b].$$

B. It turns out (see [7] for example; this is not necessary here) that if $\mathcal{B}$ is an arbitrary C*-subalgebra of $\mathcal{A}$, with $\mathcal{B} \ni I$, and such that $\mathcal{B}$ is an AW*-factor (in itself), then $\mathcal{B}$ is automatically an AW*-subfactor of $\mathcal{A}$.

Proposition 3.6. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, let $P \in \mathbf{P}(\mathcal{A})$ be a non-zero projection, let $A \in \mathcal{A}_{sa}$, and let $\mathcal{B}$ be an AW*-subfactor of $P\mathcal{A}P$, of type* II_1*. For any projection $Q \in \mathbf{P}(\mathcal{B})$ with $D_{\mathcal{A}}(Q) \geq D_{\mathcal{A}}\big(\mathbf{s}(A)\big)$, there exists $B \in (Q\mathcal{B}Q)_{sa}$ with $B \sim A$.*

Proof. By Proposition 3.6 there exists a scale $\mathcal{F} = (F, [0, \delta])$ in $\mathcal{A}$, with $F(\delta) = \mathbf{s}(A)$, and $f \in \mathfrak{R}[0, \delta]$, such that $A = \int_0^\delta f\, d\mathcal{F}$.

Denote $D_{\mathcal{A}}(P)$ simply by λ, so that
$$D_{\mathcal{B}}(Q) = D_{P\mathcal{A}P}(Q) = \frac{D_{\mathcal{A}}(Q)}{\lambda} \geq \frac{D_{\mathcal{A}}\big(\mathbf{s}(A)\big)}{\lambda} = \frac{\delta}{\lambda}.$$
Choose a projection $Q_0 \in \mathbf{P}(\mathcal{B})$ with $Q_0 \leq Q$ and $D_{\mathcal{B}}(Q_0) = \delta/\lambda$, and let $\mathcal{E} = (E, [0, \delta/\lambda])$ be a scale in $\mathcal{B}$ with $E(\delta/\lambda) = Q_0$. Let us consider the element
$$B = \int_0^{\delta/\lambda} f(\lambda t)\, dE(t)\ (\text{in } \mathcal{B}).$$
Since $\mathcal{E}$ is also a scale in $P\mathcal{A}P$, we also have the equality
$$B = \int_0^{\delta/\lambda} f(\lambda t)\, dE(t)\ (\text{in } P\mathcal{A}P).$$
Let $\mathcal{E}^P = (E^P, [0, \delta])$ be the scale in $\mathcal{A}$, constructed in Proposition 3.4. According to Proposition 3.4, we have the equality
$$B = \int_0^\delta f(t)\, dE^P(t)\ (\text{in } \mathcal{A}),$$

and then by Corollary 3.3 (applied to $f = g$ and to the scales $\mathcal{F}$ and $\mathcal{E}^P$) it follows immediately that $A \sim B$. $\square$

Comments. If $\mathcal{A}$ is an AW*-factor of type II_1, the maps ω_A, associated with elements $A \in \mathcal{A}_{sa}$, have several additional properties listed below. (These features are not needed here; see [8] for details.)

A. For any $A \in \mathcal{A}_{sa}$, the map $\omega_A : [0,1] \to \mathbb{R}$ is continuous from the left, continuous at 0, and satisfies:
$$\text{Spec}(A) = \overline{\text{Range}\, \omega_A}.$$
B. If $A \in \mathcal{A}_{sa}$ is positive, then
   (i) $\omega_A(t) = \inf \big\{ \|PAP\| \,:\, P \in \mathbf{P}(\mathcal{A}),\ D(P) \geq t \big\}, \forall\, t \in (0,1]$;
   (ii) if $\mathcal{E} = (E, [0,1])$ is a spectral scale for A, then
$$\omega_A(t) = \|E(t)A\|,\ \ \forall\, t \in (0,1].$$
C. Given a full scale $\mathcal{E}$, and a non-decreasing function $f : [0,1] \to \mathbb{R}$, which is continuous from the left, and continuous at 0, the element $A = \int_0^1 f\, d\mathcal{E} \in \mathcal{A}_{sa}$ satisfies the identity $\omega_A = f$. Moreover, $\mathcal{E}$ is a spectral scale for A.

In the case of von Neumann II_1-factors, the map $t \longmapsto \omega_A(t)$ is related to the singular numbers discussed in [9], in connection with the min-max trace formula (which is precisely property (ii) in Theorem 3.1 above, with $k = 1$). Using the language from [9], if $\mathcal{A}$ is a von Neuman II_1-factor, and $A \in \mathcal{A}$, $A \geq 0$, then for every $t \in [0,1]$, one has the equality $\omega_A(1-t) = s_t(A)$, where $s_t(A)$ is the "t^{th} singular number of A."

Example 3.1. Let $\mathcal{A}$ be an AW*-factor ot type II_1, and let $\mathcal{E} = (E, [0,1])$ be a full scale in $\mathcal{A}$. We can define the element $M = M_{\mathcal{E}} = \int_0^1 t\, dE(t) \in \mathcal{A}$. By Proposition 3.3 we know that the scalar spectral measure μ^M of M is given by
$$\int_{\mathbb{R}} \phi\, d\mu^M = \int_0^1 \phi(t)\, dt,\ \ \forall\, \phi \in C_0(\mathbb{R}). \tag{31}$$
An element $M \in \mathcal{A}_{sa}$ with property (31) is called a *mediator in* $\mathcal{A}$. The specific element $M_{\mathcal{E}}$ is referred to as the *mediator of* $\mathcal{E}$. It is obvious that $\text{Spec}(M) = [0,1]$. Given a Borel measurable function $f : [0,1] \to \mathbb{R}$, which is Riemann integrable, it is not hard to show (see [8]) that one has the equality $\int_0^1 f\, d\mathcal{E} = f(M_{\mathcal{E}})$.

We conclude with a discussion on probabilistic independence, that is necessary in the following section.

Definition. Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $\mathcal{B}$ be an AW*-subalgebra of $\mathcal{A}$. We say that $\mathcal{B}$ is *thick*, if there exist a mediator $M \in \mathcal{A}$, such that

- $BM = MB, \forall\, B \in \mathcal{B}$;
- $q_{\mathcal{A}}(BM^k) = q_{\mathcal{A}}(B) \cdot q_{\mathcal{A}}(M^k), \forall\, B \in \mathcal{B},\ k \in \mathbb{N}$.

In this case, M will be referred to as a $\mathcal{B}$-*mediator* (in $\mathcal{A}$).

Obviously the center $\mathbb{C}(= \{\lambda I \,:\, \lambda \in \mathbb{C}\})$ is thick, and every mediator is a $\mathbb{C}$-mediator. Because of possible (type) limitations on the commutant, not all AW*-subalgebras are thick. The terminology below is meant to provide a method of testing for thickness.

Definition. Two AW*-subfactors $\mathcal{B}$ and $\mathcal{M}$ of $\mathcal{A}$ are said to be *independent in probability*, if:

- $\mathcal{B}$ and $\mathcal{M}$ commute, i.e. $BM = MB$, $\forall\, B \in \mathcal{B}$, $M \in \mathcal{M}$;
- $q_{\mathcal{A}}(BM) = q_{\mathcal{B}}(B) \cdot q_{\mathcal{M}}(M)$, $\forall\, B \in \mathcal{B}_{sa}$, $M \in \mathcal{M}_{sa}$.

With this terminology, one has the following observation.

Remark 3.6. If $\mathcal{A}$ is an AW*-factor of type II_1, and $\mathcal{B}$ and $\mathcal{M}$ are two AW*-subfactors that are independent in probability, with $\mathcal{M}$ of type II_1, then $\mathcal{B}$ is thick in $\mathcal{A}$. In fact, every mediator M in $\mathcal{M}$ (such elements exist by Example 3.1) is a $\mathcal{B}$-mediator.

Example 3.2. Every AW*-factor $\mathcal{A}$, of type II_1, contains a thick subfactor of type II_1. One way to construct such subfactors is the following. We start with an AW*-subfactor $\mathcal{R} \subset \mathcal{A}$ that is isomorphic to the hyperfinite von Neumann II_1-factor (see Fact C in the introduction). Since $\mathcal{R} \simeq \mathcal{R} \otimes \mathcal{R}$ – spatial tensor product of von Neumann algebras – it follows that $\mathcal{R}$ contains two subfactors, namely $\mathcal{R} \otimes I$ and $I \otimes \mathcal{R}$, which are obviously independent in probability. Regarding these as AW*-subfactors of $\mathcal{A}$ finishes the construction.

4. Foldings

This section consists of several technical results, necessary in Section 5. At some point, a certain hypothesis (global for this section) will be set.

Definitions. Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $k \geq 1$ be an integer. A double sequence $\Phi = (A_1, \dots, A_k; B_1, \dots, B_k) \subset \mathcal{A}_{sa}$ is called a *k-folding in $\mathcal{A}$*, if:

(i) $A_i \perp B_j$, $\forall\, i, j \in \{1, \dots, k\}$;
(ii) $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ are abelian.
(iii) $A_i \sim B_i$, $\forall\, i \in \{1, \dots, k\}$.

Note that, using (i), it turns out that $\{A_1, B_1, \dots, A_k, B_k\}$ is abelian. Using (i) and (iii), it follows that the elements $S_i = A_i - B_i$, $i = 1, \dots, k$, are spectrally symmetric and they all commute. We define the *support of* Φ to be the projection

$$\mathbf{s}(\Phi) = \Big[\bigvee_{i=1}^{k} \mathbf{s}(A_i) \Big] \vee \Big[\bigvee_{i=1}^{k} \mathbf{s}(B_i) \Big]$$

We define $\|\Phi\| = \max\big\{ \|A_1\|, \|B_1\|, \dots, \|A_n\|, \|B_n\| \big\}$.

When we want to identify the elements $X = A_1 + \dots + A_k$ and $Y = B_1 + \dots + B_k$ (which are orthogonal), we are going to use the phrase: Φ *is a folding of X as Y*.

Comment. For $X, Y \in \mathcal{A}_{sa}$, the existence of a folding of X as Y, obviously implies the condition $q_{\mathcal{A}}(X) = q_{\mathcal{A}}(Y)$. The main goal of this paper is essentially to prove the converse of this statement.

Remark 4.1. Suppose $\Phi_n = (A_{n1}, \dots, A_{nk}; B_{n1}, \dots, B_{nk})$, $n = 1, \dots, N$ are k-foldings in $\mathcal{A}$, which are orthogonal, in the sense that $\mathbf{s}(\Phi_m) \perp \mathbf{s}(\Phi_n)$, $\forall\, m \neq n$. Then the double sequence $(A_1, \dots, A_k; B_1, \dots, B_k)$, defined by

$$A_j = \sum_{n=1}^{N} A_{nj} \text{ and } B_j = \sum_{n=1}^{N} B_{nj}, \ \ \forall\, j \in \{1, \dots, k\},$$

is a k-folding, which will be denoted by $\Phi_1 + \dots + \Phi_n$. This follows essentially from Corollary 2.1. It is also pretty clear that $\|\Phi_1 + \dots + \Phi_n\| = \max\big\{ \|\Phi_1\|, \dots, \|\Phi_n\| \big\}$.

In what follows, we are going to isolate a special type of 2-foldings, that consist of projections.

Definitions. A *superprojection in $\mathcal{A}$* is a system $\pi = (P_1, P_2; P_3, P_4)$ of projections in $\mathcal{A}$, with the following properties:

- $P_i \perp P_j$, $\forall\, i \neq j$;
- $P_1 \sim P_3$ and $P_2 \sim P_4$.

It is obvious that π is a 2-folding.

Given another superprojection $\pi' = (P_1', P_2'; P_3', P_4')$, we write $\pi' \leq \pi$, if $P_i' \leq P_i$, $i = 1, 2, 3, 4$.

Lemma 4.1. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, let $\pi = (P_1, P_2; P_3, P_4)$ be a superprojection in $\mathcal{A}$, and let $\alpha, \beta > 0$ be two real numbers with the property:*

$$\alpha D(P_1) = \beta D(P_2). \tag{32}$$

If, for each $k = 1, \dots, 4$, a mediator M_k in $P_k \mathcal{A} P_k$ is given, then the system $\Gamma = (A, B; V, W)$, defined by

$$A = \alpha P_1 + \beta M_1 + \alpha M_4; \qquad B = -\beta M_1 - \alpha M_4;$$
$$V = \beta P_2 + \alpha M_2 + \beta M_3; \qquad W = -\beta M_3 - \alpha M_2.$$

is a 2-folding of αP_1 as βP_2.

Proof. Consider the numbers $a = D(P_2)$ and $b = D(P_1)$, and denote the common value $a/\alpha = b/\beta$ by λ. Since, for every $j \in \{1, \dots, 4\}$ we have

$$q_{P_j \mathcal{A} P_j}(X) = \frac{q_{\mathcal{A}}(X)}{D(P_j)}, \ \ \forall\, X \in P_j \mathcal{A} P_j,$$

we get the equalities

$$q_{\mathcal{A}}(M_1^k) = q_{\mathcal{A}}(M_3^k) = \frac{b}{k+1}, \tag{33}$$

$$q_{\mathcal{A}}(M_2^k) = q_{\mathcal{A}}(M_4^k) = \frac{a}{k+1}, \tag{34}$$

for all integers $k \geq 1$.

Note that, since $P_1, \dots, P_4, M_1, \dots, M_4$ commute, the elements A, B, V, W also commute.

Using the obvious orthogonality relations $(\alpha P_1 + \beta M_1) \perp (\alpha M_4)$, $(\beta M_1) \perp (\alpha M_4)$, $(\beta P_2 + \alpha M_2) \perp (\beta M_3)$, and $(\alpha M_2) \perp (\beta M_3)$, one has the the equalities:

$$A^k = \alpha^k M_4^k + (\alpha P_1 + \beta M_1)^k = \alpha^k \big[M_4^k + P_1\big] + \sum_{j=1}^{k} \binom{k}{j} \alpha^{k-j} \beta^j M_1^j, \tag{35}$$

$$B^k = (-1)^k \big[\beta^k M_1^k + \alpha^k M_4^k\big], \tag{36}$$

$$V^k = \beta^k M_3^k + (\beta P_2 + \alpha M_2)^k = \beta^k \big[M_3^k + P_2\big] + \sum_{j=1}^{k} \binom{k}{j} \beta^{k-j} \alpha^j M_2^j, \tag{37}$$

$$W^k = (-1)^k \big[\alpha^k M_2^k + \beta^k M_3^k\big], \tag{38}$$

for every integer $k \geq 1$. It is also pretty obvious that $A, B \perp V, W$, and we have the equalities $A + B = \alpha P_1$ and $V + W = \beta P_2$, so in order to finish the proof, we are left to show that $A \sim V$, and $B \sim W$. For this purpose we use Theorem 2.1, which means that it suffices to prove the equalities $q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(V^k)$ and

$q_{\mathcal{A}}(B^k) = q_{\mathcal{A}}(W^k)$, $\forall\, k \in \mathbb{N}$. These equalities are proven by direct computation, as follows.

For B and W the equality follows from (33) and (34), which immediately give:

$$q_{\mathcal{A}}(B^k) = q_{\mathcal{A}}(W^k) = \frac{(-1)^k[\alpha^k a + \beta^k b]}{k+1}.$$

For A and V, again using (33) and (34), we have

$$\begin{aligned}
q_{\mathcal{A}}(A^k) &= \alpha^k\big[q_{\mathcal{A}}(M_4^k) + b\big] + \sum_{j=1}^{k}\binom{k}{j}\alpha^{k-j}\beta^j q_{\mathcal{A}}(M_1^j) = \\
&= \frac{\alpha^k a}{k+1} + b\Big[\alpha^k + \sum_{j=1}^{k}\binom{k}{j}\frac{\alpha^{k-j}\beta^j}{j+1}\Big] = \frac{\alpha^k a}{k+1} + b\Big[\sum_{j=0}^{k}\binom{k}{j}\frac{\alpha^{k-j}\beta^j}{j+1}\Big]; \\
q_{\mathcal{A}}(V^k) &= \beta^k\big[q_{\mathcal{A}}(M_3^k) + a\big] + \sum_{j=1}^{k}\binom{k}{j}\beta^{k-j}\alpha^j q_{\mathcal{A}}(M_2^j) = \\
&= \frac{\beta^k b}{k+1} + a\Big[\beta^k + \sum_{j=1}^{k}\binom{k}{j}\frac{\beta^{k-j}\alpha^j}{j+1}\Big] = \frac{\beta^k b}{k+1} + a\Big[\sum_{j=0}^{k}\binom{k}{j}\frac{\beta^{k-j}\alpha^j}{j+1}\Big].
\end{aligned}$$

Replacing $a = \lambda\alpha$ and $b = \beta\lambda$, the above computations continue as

$$\begin{aligned}
q_{\mathcal{A}}(A^k) &= \lambda\Big(\frac{\alpha^{k+1}}{k+1} + \sum_{j=0}^{k}\binom{k}{j}\frac{\alpha^{k-j}\beta^{j+1}}{j+1}\Big) = \\
&= \lambda\Big(\frac{\alpha^{k+1}}{k+1} + \int_0^\beta \big[\sum_{j=0}^{k}\binom{k}{j}\alpha^{k-j}t^j\big]\,dt\Big) = \\
&= \lambda\Big(\frac{\alpha^{k+1}}{k+1} + \int_0^\beta (\alpha+t)^k\,dt\Big) = \frac{\lambda(\alpha+\beta)^{k+1}}{k+1}, \\
q_{\mathcal{A}}(V^k) &= \lambda\Big(\frac{\beta^{k+1}}{k+1} + \sum_{j=0}^{k}\binom{k}{j}\frac{\beta^{k-j}\alpha^{j+1}}{j+1}\Big) = \\
&= \lambda\Big(\frac{\beta^{k+1}}{k+1} + \int_0^\alpha \big[\sum_{j=0}^{k}\binom{k}{j}\beta^{k-j}t^j\big]\,dt\Big) = \\
&= \lambda\Big(\frac{\beta^{k+1}}{k+1} + \int_0^\alpha (\beta+t)^k\,dt\Big) = \frac{\lambda(\alpha+\beta)^{k+1}}{k+1},
\end{aligned}$$

so we indeed have the equality $q_{\mathcal{A}}(A^k) = q_{\mathcal{A}}(V^k)$. $\square$

Convention. For the remainder of this section we are going to work under the following assumptions: *We fix $\mathcal{A}$ to be an AW*-factor of type* II_1. *We fix a thick AW*-subfactor $\mathcal{B}$ of type* II_1 *(which exists by Example 3.2). We fix a $\mathcal{B}$-mediator M in $\mathcal{A}$.*

Notation. Let $\pi = (P, Q; P', Q') \in \mathbf{\Pi}(\mathcal{B})$, and let α, β be positive real numbers. We define the system $\Gamma^{\alpha\beta}(\pi) = (A, B; V, W) \subset \mathcal{A}_{sa}$ by:

$$\begin{aligned}
A &= P(\alpha I + \beta M) + \alpha Q'M; & B &= -\beta PM - \alpha Q'M; \\
V &= Q(\beta I + \alpha M) + \beta P'M; & W &= -\beta P'M - \alpha QM.
\end{aligned}$$

Proposition 4.1. *If $\pi = (P, Q; P', Q') \in \mathbf{\Pi}(\mathcal{B})$, and if the real numbers $\alpha, \beta > 0$ satisfy the condition:*

$$\alpha D(P) = \beta D(Q), \tag{39}$$

then $\Gamma^{\alpha\beta}(\pi)$ is a 2-folding of αP as βQ.

Proof. By Lemma 4.1, all we must show is the fact that PM is a mediator in $P\mathcal{A}P$, $P'M$ is a mediator in $P'\mathcal{A}P'$, QM is a mediator in $Q\mathcal{A}Q$, and $Q'M$ is a mediator in $Q'\mathcal{A}Q$. But this is obvious, since P, P', Q, Q' all belong to $\mathcal{B}$, and M is a $\mathcal{B}$-mediator. ☐

For the purpose of a smooth exposition, we isolate the hypothesis of the above result as follows.

Definition. Given a superprojection $\pi = (P, Q; P', Q') \in \mathbf{\Pi}(\mathcal{B})$, and two real numbers $\alpha, \beta > 0$, we declare π to be of *of type $\alpha|\beta$* – or say π is an *$\alpha|\beta$-superprojection* – if π satisfies condition (39). (The reason we use the notation $\alpha|\beta$ is the fact that the feature we are interested in does not change if both α and β are multiplied by a factor.)

Theorem 4.1 (Local Folding)**.** *Let $P, Q \in \mathbf{P}(\mathcal{B})$ be two projections with $Q \leq P$, let $X \in P\mathcal{B}P$, be a positive element, and let $\beta > 0$ be a real number with the following properties*

- (i) *$X \perp Q$;*
- (ii) *$q_{\mathcal{A}}(X) = \beta D(Q)$;*
- (iii) *$D(P) \geq 2\big[D\big(\mathbf{s}(X)\big) + D(Q)\big]$.*

Then there exists a 2-folding Φ, of X as βQ, with $\mathbf{s}(\Phi) \leq P$.

Proof. We begin the proof by fixing some notations.

Denote for simplicity $\mathbf{s}(X)$ by S, and $D(S)$ by δ. Use Proposition 3.6 to find a scale $\mathcal{E} = (E, [0, \delta])$ in $\mathcal{B}$ with $E(\delta) = S$, and a non-decreasing function $f : [0, \delta] \to \big[0, \|X\|\big]$ such that $X = \int_0^\delta f\, d\mathcal{E}$.

Since $S \perp Q$, we know that $S + Q \leq P$, and moreover $D(P) \geq 2\big[D(S) + D(Q)\big]$. In particular, there exist two more projections $S', Q' \in \mathbf{P}(\mathcal{B})$, with $S', Q' \leq P$, such that $\sigma = (S, Q; S', Q')$ is a superprojection. Let then $\mathcal{E}' = (E', [0, \delta])$ be a scale in $\mathcal{B}$, with $E'(\delta) = S'$, and define the element $X' = \int_0^\delta f\, d\mathcal{E}'$.

Let us define, for any closed subinterval $J = [a, b]$ of $[0, \delta]$, the number $\alpha_J = \inf_{t \in J} f(t)$ (note that $0 \leq \alpha_J \leq \|X\|$), and the projections $E_J = E(b) - E(a)$, and $E'_J = E'(b) - E'(a)$. (Of course, both E_J and E'_J belong to $\mathcal{B}$, they are orthogonal – since $E_J \leq R$ and $E'_J \leq R'$, and they are equivalent, since $D(E_J) = D(E'_J) = b - a$.)

We also fix a sequence of partitions $(\Delta_n)_{n=1}^\infty \subset \mathfrak{P}[0, \delta]$, such that

- (a) $\Delta_1 \subset \Delta_2 \subset \dots$;
- (b) $\lim_{n \to \infty} L(f, \Delta_n) = \int_0^\infty f(t)\, dt$.

In fact, we can also assume that $\Delta_1 = [0 < \delta]$, and

- (c) *for every $n \geq 1$, the partition Δ_{n+1} is obtained by subdividing exactly one interval in Δ_n into two sub-intervals.*

In other words, if $\Delta_n = [0 = t_0 < t_1 < \dots < t_n = \delta]$, then $\Delta_{n+1} = [0 = s_0 < s_1 \dots < s_{n+1} = \delta]$, with $\{t_0, t_1, \dots, t_n\} \subset \{s_0, s_1, \dots, s_{n+1}\}$. (The fact that Δ_n consists of a partition into n intervals is no coincidence.)

For every $n \geq 1$, let $\mathcal{J}_n$ be the set of intervals determined by Δ_n. (Namely, if $\Delta_n = [0 = t_0 < t_1 < \cdots < t_n = \delta]$, then $\mathcal{J}_n = \big\{[t_{i-1}, t_i] \,:\, i = 1, \dots, n\big\}$.) With this notation, $\mathcal{J}_{n+1}$ is obtained from $\mathcal{J}_n$ by splitting (exactly) one of its intervals – denoted J_n – into two sub-intervals, denoted L_n (the left one) and R_n (the right one), so if, say $J_n = [a, b]$, then $L_n = [a, c]$ and $R_n = [c, b]$ for some $a < c < b$. With this notation, we have: $\mathcal{J}_{n+1} = (\mathcal{J}_n \smallsetminus \{J_n\}) \cup \{L_n, R_n\}$.

Denote for simplicity $L_{\mathcal{E}}(f, \Delta_n)$ by X_n, and $L_{\mathcal{E}'}(f, \Delta_n)$ by X'_n. With these notations, one obviously has the equalities

$$X_n = \sum_{J \in \mathcal{J}_n} \alpha_J E_J \text{ and } X'_n = \sum_{J \in \mathcal{J}_n} \alpha_J E'_J, \;\; \forall\, n \in \mathbb{N}, \tag{40}$$

with $0 \leq X_n \leq X$ and $0 \leq X'_n \leq X'$.

Claim 1. Let $\mathcal{J} = \bigcup_{n=1}^{\infty} \mathcal{J}_n$. *There exist two maps* $\mathcal{J} \ni J \longmapsto Q_J \in \mathbf{P}(\mathcal{B})$ *and* $\mathcal{J} \ni J \longmapsto Q'_J \in \mathbf{P}(\mathcal{B})$, *with the following properties:*

- (A) $Q_J \leq Q$ *and* $Q'_J \leq Q'$ *(hence* $Q_J \perp Q'_J$*)*, $\forall\, J \in \mathcal{J}$*;*
- (B) $Q_J \sim Q'_J$, *and* $\alpha_J D(E_J) = \beta D(Q_J)$, $\forall\, J \in \mathcal{J}$*;*
- (C) *if* $J, K \in \mathcal{J}$ *are essentially disjoint (i.e.* $J \cap K$ *has at most one point), then* $Q_J \perp Q_K$ *and* $Q'_J \perp Q'_K$*;*
- (D) $Q_{J_n} \geq Q_{L_n}$ *and* $Q'_{J_n} \geq Q'_{L_n}$, $\forall\, n \geq 1$*;*
- (E) $Q_{L_n} + Q_{R_n} \geq Q_{J_n}$ *and* $Q'_{L_n} + Q'_{R_n} \geq Q'_{J_n}$, $\forall\, n \geq 1$.

The two maps will be defined recursively. Put $\tilde{\mathcal{J}}_k = \mathcal{J}_{k-1} \cup \mathcal{J}_k$ (with the convention $\mathcal{J}_0 = \varnothing$), so that we still have $\mathcal{J} = \bigcup_{k=1}^{\infty} \tilde{\mathcal{J}}_k$, and $\tilde{\mathcal{J}}_{k+1} = \tilde{\mathcal{J}}_k \cup \{L_k, R_k\}$ (disjoint union). Of course, $\tilde{\mathcal{J}}_1 = \mathcal{J}_1 = \{J_1\}$, where $J_1 = [0, \delta]$. Start off by defining Q_{J_1} and Q'_{J_1} to be arbitrary projections with $Q_{J_1} \leq Q$ and $Q'_{J_1} \leq Q'$, such that

$$D(Q_{J_1}) = D(Q'_{J_1}) = \frac{\alpha_{J_1} D(E_{J_1})}{\beta}.$$

This is possible, since

$$\frac{\alpha_{J_1} D(E_{J_1})}{\beta} = \frac{q_{\mathcal{A}}(X_1)}{\beta} \leq \frac{q_{\mathcal{A}}(X)}{\beta} = D(Q) = D(Q').$$

Assume now the projections Q_J and Q'_J are defined for all $J \in \tilde{\mathcal{J}}_k$, and they satisfy conditions (A)(B)(C) (with $\tilde{\mathcal{J}}_k$ in place of $\mathcal{J}$), and conditions (D) and (E) for all $n < k$, and let us indicate how to construct the “new” projections Q_{L_k}, Q'_{L_k}, Q_{R_k}, and Q'_{R_k}.

Define the elements $F_k = \sum_{J \in \mathcal{J}_k} Q_J$ and $F'_k = \sum_{J \in \mathcal{J}_k} Q'_J$ in $\mathcal{B}$. First of all, using (B) (C), it follows that F_k and F'_k are projections in $\mathcal{B}$, with $F_k \leq Q$, $F'_k \leq Q'$, and

$$\beta D(F_k) = \beta D(F'_k) = \sum_{J \in \mathcal{J}_k} \alpha_J D(E_J) = q_{\mathcal{A}}(X_k). \tag{41}$$

Secondly, since f is non-decreasing, we have:

$$\alpha_{L_k} = \alpha_{J_k} \text{ and } \alpha_{R_k} \geq \alpha_{J_k}. \tag{42}$$

Using (B) for $J = J_k$, it follows that

$$\frac{\alpha_{L_k} D(E_{L_k})}{\beta} = \frac{\alpha_{J_k} D(E_{L_k})}{\beta} \leq \frac{\alpha_{J_k} D(E_{J_k})}{\beta} = D(Q_{J_k}) = D(Q'_{J_k}),$$

so we can choose two projections $Q_{L_k}, Q'_{L_k} \in \mathbf{P}(\mathcal{B})$, with $Q_{L_k} \leq Q_{J_k}$, $Q'_{L_k} \leq Q'_{J_k}$, and

$$D(Q_{L_k}) = D(Q'_{L_k}) = \frac{\alpha_{L_k} D(E_{L_k})}{\beta}. \tag{43}$$

Let us observe that, using (41) and the obvious equality

$$X_{k+1} - X_k = [\alpha_{R_k} - \alpha_{J_k}] \cdot E_{R_k},$$

one has (all terms commute):

$$\begin{aligned} [\alpha_{R_k} - \alpha_{J_k}] \cdot D(E_{R_k}) = q_{\mathcal{A}}(X_{k+1} - X_k) = q_{\mathcal{A}}(X_{k+1}) - q_{\mathcal{A}}(X_k) \leq \\ \leq q_{\mathcal{A}}(X) - q_{\mathcal{A}}(X_k) = \beta D(Q) - \beta D(F_k) = \beta D(Q - F_k). \end{aligned}$$

In particular, one has the inequalities

$$D(Q - F_k) = D(Q' - F'_k) \geq \frac{[\alpha_{R_k} - \alpha_{J_k}] \cdot D(E_{R_k})}{\beta},$$

so there exist projections $G, G' \in \mathbf{P}(\mathcal{B})$, with $G \leq Q - F_k$, $G' \leq Q' - F'_k$, such that

$$D(G) = D(G') = \frac{[\alpha_{R_k} - \alpha_{J_k}] \cdot D(E_{R_k})}{\beta}.$$

By construction, we have

$$G \perp F_k \text{ and } G' \perp F'_k. \tag{44}$$

Since $0 \leq Q_{J_k} - Q_{L_k} \leq Q_{J_k} \leq F_k$ and $0 \leq Q'_{J_k} - Q'_{L_k} \leq Q'_{J_k} \leq F'_k$, using (44), we have $G \perp (Q_{J_k} - Q_{L_k})$ and $G' \perp (Q'_{J_k} - Q'_{L_k})$. We can then define then the projections $Q_{R_k} = G + (Q_{J_k} - Q_{L_k})$ and $Q'_{R_k} = G' + (Q'_{J_k} - Q'_{L_k})$.

We now check conditions (A)(B)(C) with $\tilde{\mathcal{J}}_{k+1}$ in place of $\mathcal{J}$, and conditions (D)(E) with $n = k$.

To check condition (A), we only need to consider the "new" intervals, namely the cases $J = L_k, R_k$, which are obvious by construction.

To check condition (B) we need to prove the equalities

$$\alpha_{L_k} D(E_{L_k}) = \beta D(Q_{L_k}) = \beta D(Q'_{L_k}), \tag{45}$$

$$\alpha_{R_k} D(E_{R_k}) = \beta D(Q_{R_k}) = \beta D(Q'_{R_k}). \tag{46}$$

The equalities (45) are trivial using (43). To prove the equalities (46), we first notice that by construction we have

$$\begin{aligned} D(Q_{R_k}) = D(G) + D(Q_{J_k} - Q_{L_k}) = D(G) + D(Q_{J_k}) - D(Q_{L_k}) = \\ = \frac{[\alpha_{R_k} - \alpha_{J_k}] \cdot D(E_{R_k}) + \alpha_{J_k} D(E_{J_k}) - \alpha_{L_k} D(E_{L_k})}{\beta}. \end{aligned}$$

Using the equalities $\alpha_{L_k} = \alpha_{J_k}$, as well as $D(E_{J_k}) = D(E_{L_k}) + D(E_{R_k})$, the above computation continues as

$$\begin{aligned} D(Q_{R_k}) = \frac{[\alpha_{R_k} - \alpha_{J_k}] \cdot D(E_{R_k}) + \alpha_{J_k} D(E_{J_k}) - \alpha_{J_k} \cdot \big[D(E_{J_k}) - D(E_{L_k})\big]}{\beta} = \\ = \frac{\alpha_{R_k} D(E_{R_k})}{\beta} = \dots \text{ (same computation for } Q'_{R_k}) \ \dots = D(Q'_{R_k}), \end{aligned}$$

thus proving (46).

To prove condition (C) we only need to examine the "new" cases, which are

$$Q_{L_k} \perp Q_{R_k}, \text{ and } Q'_{L_k} \perp Q'_{R_k}; \tag{47}$$

$$Q_{L_k} \perp Q_J, \text{ and } Q'_{L_k} \perp Q'_J, \ \ \forall\, J \in \mathcal{J}_k \smallsetminus \{J_k\}.; \tag{48}$$

$$Q_{R_k} \perp Q_J, \text{ and } Q'_{R_k} \perp Q'_J, \ \ \forall\, J \in \mathcal{J}_k \smallsetminus \{J_k\}.. \tag{49}$$

The orthogonality relations (47) follow from (44), which together with the obvious inequalities $Q_{L_k} \leq Q_{J_k} \leq F_k$ and $Q'_{L_k} \leq Q'_{J_k} \leq F'_k$ force $G \perp Q_{L_k}$ and $G' \perp Q'_{L_k}$. To prove (48) and (49) we simply observe that

$$Q_{L_k} + Q_{R_k} = Q_{J_k} + G \text{ and } Q'_{L_k} + Q'_{R_k} = Q_{J_k} + G,$$

and by (44) we also have

$$G \perp Q_J \text{ and } G' \perp Q'_J, \ \ \forall\, J \in \mathcal{J}_k.$$

Conditions (D)(E) – for $k = n$ – are automatically satisfied, by construction.

Having proven Claim 1, we continue the proof of the Theorem, by fixing the maps $(Q_J)_{J\in\mathcal{J}}$ and $(Q'_J)_{J\in\mathcal{J}}$ as above. It is obvious that, for every $J \in \mathcal{J}$, the system $\pi_J = (E_J, Q_J; E'_J, Q'_J)$ is an $\alpha_J|\beta$-superprojection. (By construction $\pi_J \leq \sigma$.) Use Proposition 4.1 to define, for each $J \in \mathcal{J}$, a 2-folding $\Gamma_J = (A^J, B^J; V^J, W^J)$, by

$$A^J = E_J(\alpha_J I + \beta M) + \alpha_J Q'_J M; \qquad B^J = -\beta E_J M - \alpha_J Q'_J M;$$
$$V^J = Q_J(\beta I + \alpha_J M) + \beta E'_J M; \qquad W^J = -\beta E'_J M - \alpha_J Q_J M.$$

Claim 2. The 2*-foldings* $(\Gamma_J)_{J\in\mathcal{J}}$ *have the following properties:*

- $\|\Gamma_J\| \leq 2\big[\beta + \|X\|\big]$, $\forall\, J \in \mathcal{J}$.
- *if* $J, K \in \mathcal{J}$ *are essentially disjoint, then* $\mathbf{s}(\Gamma_J) \perp \mathbf{s}(\Gamma_K)$*;*

The first assertion is trivial, since $0 \leq \alpha_J \leq \|X\|$, $\forall\, J \in \mathcal{J}$.

To prove the second property, we start with two intervals $J, K \in \mathcal{J}$ that are essentially disjoint, and we notice that the collection $\{E_J, E'_J, E_K, E'_K, Q_J, Q'_J, Q_K, Q'_K\}$ is orthogonal. (This can be done by "groupping," observing that, since we have $E_J, E_K \leq S$, $E_J, E_K \leq S'$, $Q_J, Q_K \leq Q$, and $Q'_J, Q'_K \leq Q$, with $\{S, S', Q, Q'\}$ orthogonal, all we must show are the orthogonality relations: $E_J \perp E_K$, $E'_J \perp E_K$, $Q_J \perp Q_K$, and $Q'_J \perp Q'_K$, which are obvious.) The first stament is then clear, since we have the following (obvious) inequalities:

$$\mathbf{s}(\Gamma^J) \leq E_J + E'_J + Q_J + Q'_J,$$
$$\mathbf{s}(\Gamma^K) \leq E_K + E'_K + Q_K + Q'_K,$$

with $(E_J + E'_J + Q_J + Q'_J) \perp (E_K + E'_K + Q_K + Q'_K)$.

Having proven Claim 2, let us define now, for every integer $n \geq 1$, the system $\Phi_n = (A_n, B_n; V_n, W_n)$, where $A_n = \sum_{J\in\mathcal{J}_n} A^J$, $B_n = \sum_{J\in\mathcal{J}_n} B^J$, $V_n = \sum_{J\in\mathcal{J}_n} V^J$, and $W_n = \sum_{J\in\mathcal{J}_n} W^J$.

Claim 3. For every $n \in \mathbb{N}$*, the system* Φ_n *is a* 2*-folding of* X_n *as* βF_n*, where* $F_n = \sum_{J\in\mathcal{J}_n} Q_J$*. Moreover, one has* $\|\Phi_n\| \leq 2\big[\beta + \|X\|\big]$, $\forall\, n \in \mathbb{N}$.

This follows from Remark 4.1, combined with Claim 2, and the fact that all the intervals in $\mathcal{J}_n$ are essentially disjoint. By construction, we have

$$A_n + B_n = \sum_{J\in\mathcal{J}_n} \alpha_J E_J = X_n,$$
$$V_n + W_n = \beta \sum_{J\in\mathcal{J}_n} Q_J = \beta F_n,$$

so Φ_n is indeed a 2-folding of X_n as βF_n.

Claim 4. The sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$, and $(W_n)_{n\in\mathbb{N}}$ have the following properties:

- *they are all bounded;*
- *they are jointly abelian;*
- *they all all lie in $(P\mathcal{A}P)_{sa}$; more precisely, for every $n \in \mathbb{N}$, one has the inequalities:*

$$\mathbf{s}(A_n) \le S + Q', \quad \mathbf{s}(B_n) \le S + Q', \tag{50}$$
$$\mathbf{s}(V_n) \le S' + Q, \quad \mathbf{s}(W_n) \le S' + Q. \tag{51}$$

Moreover:

- *the sequences $(A_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ are non-decreasing;*
- *the sequences $(B_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are non-increasing.*

The first assertion is quite clear. The inequalities (50) (51) are also clear, and they imply the third assertion. The fact that the four sequences are jointly abelian follows from the fact that the collection

$$\mathcal{C} = \mathcal{E} \cup \mathcal{E}' \cup \{M\} \cup \{Q_J : J \in \mathcal{J}\} \cup \{Q'_J : J \in \mathcal{J}\} \subset (P\mathcal{A}P)_{sa} \tag{52}$$

is abelian, and obviously all sequences lie in the abelian von Neumann subalgebra $\mathcal{N} = \mathcal{C}''$.

We proceed now with the proof of the monotonicity features. Fix some $k \in \mathbb{N}$, and let us compare A_{k+1} with A_k, B_{k+1} with B_n, V_{k+1} with V_k, and W_{k+1} with W_k. Since $\mathcal{J}_k = (\mathcal{J}_n \smallsetminus \{J_k\}) \cup \{L_k, R_k\}$, we have

$$\begin{aligned}
A_{k+1} - A_k &= A^{L_k} + A^{R_k} - A^{J_k} = \big[E_{L_k}(\alpha_{L_k} I + \beta M) + \alpha_{L_k} Q'_{L_k} M\big] + \\
&\quad + \big[E_{R_k}(\alpha_{R_k} I + \beta M) + \alpha_{R_k} Q'_{R_k} M\big] - \big[E_{J_k}(\alpha_{J_k} I + \beta M) + \alpha_{J_k} Q'_{J_n} M\big];\\
B_k - B_{k+1} &= B^{J_k} - B^{L_k} - B^{R_k} = \big[\beta E'_{L_k} M + \alpha_{L_k} Q'_{L_k} M\big] + \\
&\quad + \big[\beta E'_{R_k} M + \alpha_{R_k} Q'_{R_k} M\big] - \big[\beta E'_{J_k} M + \alpha_{J_k} Q'_{J_k} M\big];\\
V_{k+1} - V_k &= V^{L_k} + V^{R_k} - V^{J_k} = \big[Q_{L_k}(\beta I + \alpha_{L_k} M) + \beta E'_{L_k} M\big] + \\
&\quad + \big[Q_{R_k}(\beta I + \alpha_{R_k} M) + \beta E'_{R_k} M\big] - \big[Q_{J_k}(\beta I + \alpha_{J_k} M) + \beta E'_{J_k} M\big];\\
W_k - W_{k+1} &= W^{J_k} - W^{L_k} - W^{R_k} = \big[\beta E'_{L_k} M + \alpha_{L_k} Q_{L_k} M\big] + \\
&\quad + \big[\beta E'_{R_k} M + \alpha_{R_k} Q_{R_k} M\big] - \big[\beta E'_{J_k} M + \alpha_{J_k} Q_{J_k} M\big].
\end{aligned}$$

Using (42) and property (E) from Claim 1, combined with the equalities $E_{J_k} = E_{J_k^\ell} + E_{J_k^r}$ and $E'_{J_k} = E'_{J_k^\ell} + E'_{J_k^r}$, we can continue with:

$$
\begin{aligned}
&A_{k+1} - A_k = [\alpha_{R_k} - \alpha_{J_k}]E_{R_k} + [\alpha_{J_k} Q_{L_k} + \alpha_{R_k} Q_{R_k} - \alpha_{J_k} Q_{J_k}]M \geq \\
&\qquad \geq \alpha_{J_k}[Q_{L_k} + Q_{R_k} - Q_{J_k}]M \geq 0; \\
&B_k - B_{k+1} = [\alpha_{J_k} Q'_{L_k} + \alpha_{R_k} Q'_{R_k} - \alpha_{J_k} Q'_{J_k}]M \geq \alpha_{J_k}[Q'_{L_k} + Q'_{R_k} - Q'_{J_k}]M \geq 0; \\
&V_{k+1} - V_k = \beta[Q_{L_k} + Q_{R_k} - Q_{J_k}] + [\alpha_{J_k} Q_{L_k} + \alpha_{R_k} Q_{R_k} - \alpha_{J_k} Q_{J_k}]M \geq \\
&\qquad \geq \alpha_{J_k}[Q_{L_k} + Q_{R_k} - Q_{J_k}]M \geq 0; \\
&W_k - W_{k+1} = [\alpha_{J_k} Q_{L_k} + \alpha_{R_k} Q_{R_k} - \alpha_{J_k} Q_{J_k}]M \geq \alpha_{J_k}[Q_{L_k} + Q_{R_k} - Q_{J_k}]M \geq 0.
\end{aligned}
$$

Having proven Claim 4, we now use Lemma 1.1, which gives the existence of the weak limits $A = \mathsf{W}\text{-}\lim_{n\to\infty} A_n$, $B = \mathsf{W}\text{-}\lim_{n\to\infty} B_n$, $V = \mathsf{W}\text{-}\lim_{n\to\infty} V_n$, and $W = \mathsf{W}\text{-}\lim_{n\to\infty} W_n$. The proof of the Theorem will be finished, once we prove the following.

Claim 5. The system $\Phi = (A, B; V, W)$ is a 2-folding of X as βQ, with $\mathbf{s}(\Phi) \leq P$.

First of all, since all four sequences lie in the abelian von Neumann algebra $\mathcal{N} = \mathcal{C}''$, with $\mathcal{C}$ defined by (52), it follows that A, B, V, and W all belong to $\mathcal{N}$. In particular, these four self-adjoint elements commute.

Note also that, since all sequences lie in $P\mathcal{A}P$, it follows that $A, B, V, W \in P\mathcal{A}P$.

Using Proposition 2.3, in conjunction with Claim 3, it is obvious that $A \sim V$ and $B \sim W$.

Moreover, by Remark 1.4 and by Claim 3, we also have the equalities

$$A + B = \underset{n\to\infty}{\mathsf{W}\text{-}\lim}(A_n + B_n) = \underset{n\to\infty}{\mathsf{W}\text{-}\lim}\, X_n = X; \tag{53}$$

$$V + W = \underset{n\to\infty}{\mathsf{W}\text{-}\lim}(V_n + W_n) = \beta\, \underset{n\to\infty}{\mathsf{W}\text{-}\lim}\, F_n. \tag{54}$$

Since

$$F_{k+1} - F_k = Q^{L_k} + Q^{R_k} - Q_{J_k} \geq 0, \ \ \forall\, k \in \mathbb{N},$$

we know that $\mathsf{W}\text{-}\lim_{n\to\infty} F_n$ is in fact a projection. Moreover, since we also know that $F_n \leq Q$, i.e. $F_n \in Q\mathcal{A}Q$, $\forall\, n \in \mathbb{N}$, it follows that $F \in Q\mathcal{A}Q$, so $F \leq Q$. By (53) and (54), combined with Lemma 1.1, we know that

$$
\begin{aligned}
\beta D(F) &= \lim_{n\to\infty} q_{\mathcal{A}}(\beta F_n) = \lim_{n\to\infty} \big[q_{\mathcal{A}}(V_n) + q_{\mathcal{A}}(W_n)\big] = \\
&= \lim_{n\to\infty} \big[q_{\mathcal{A}}(A_n) + q_{\mathcal{A}}(B_n)\big] = \lim_{n\to\infty} q_{\mathcal{A}}(X_n) = q_{\mathcal{A}}(X).
\end{aligned}
$$

This forces, of course $\beta D(F) = \beta D(Q)$, and then the condition $F \leq Q$ (combined with $\beta > 0$) will force $F = Q$.

At this point the only properties left to be proven are the orthogonality relations $A \perp V$, $A \perp W$, $B \perp V$, and $B \perp W$. For this purpose we use (50) and (51), to conclude that $A, B \in (S + Q')\mathcal{A}(S + Q')$ and $V, W \in (S' + Q)\mathcal{A}(S + Q')$, and then everything follows from $(S + Q') \perp (S' + Q)$. □

5. Self-adjoint elements with zero quasitrace

In this section we prove the main results of this paper.

Theorem 5.1. *Let $\mathcal{A}$ be an AW*factor of type II_1, and let $X \in \mathcal{A}_{sa}$ be an element with $q_{\mathcal{A}}(X) = 0$. Assume:*

(i) $D\big(\mathbf{s}(X)\big) < \frac{1}{2}$;
(ii) *there is a thick AW*-subfactor $\mathcal{B}$, of type* II_1*, which contains X.*

Then there exist a spectrally symmetric element $S \in \mathcal{B}$ and a 2-folding of X as S.

Proof. We assume of course that $X \neq 0$ (otherwise we can take $X_1 = X_2 = S = 0$).

Let $E_1 = \mathbf{s}(X^+)$ and $E_2 = \mathbf{s}(X^-)$, so that $E_1, E_2 \in \mathbf{P}(\mathcal{B})$, are orthogonal, and $\mathbf{s}(X) = E_1 + E_2$. By condition (ii) we know that $D(E_1) + D(E_2) < \frac{1}{2}$. Chose then six more projections $E_3, E_4, Q_1, Q_2, Q_3, Q_4 \in \mathbf{P}(\mathcal{B})$, such that

(A) all eight projections $E_1, \dots, E_4, Q_1, \dots, Q_4$ are orthogonal;
(B) $E_1 + \dots + E_4 + Q_1 + \dots + Q_4 = I$;
(C) $E_1 \sim E_3$ and $E_2 \sim E_4$;
(D) $Q_1 \sim Q_2 \sim Q_3 \sim Q_4$.

Denote $D(Q_1)$ simply by δ, so that $D(Q_k) = \delta$, $k = 1, \dots, 4$.

Since we are assuming $X \neq 0$, and $q_{\mathcal{A}}(X) = 0$, it follows that

$$q_{\mathcal{A}}(X^+) = q_{\mathcal{A}}(X^-) > 0.$$

Denote this common value by α, and let $\beta = \alpha/\delta$.

Consider now the projections $P_1 = E_1 + E_3 + Q_1 + Q_2$ and $P_2 = I - P_1 = E_2 + E_4 + Q_3 + Q_4$, which satisfy:

$$D(P_1) = 2\big[D(E_1) + D(Q_1)\big]; \tag{55}$$

$$D(P_2) = 2\big[D(E_2) + D(Q_3)\big]; \tag{56}$$

Remark that using the above two equalities we have the following situations.

(I) Since $\mathbf{s}(X^+) = E_1$, it follows that:
- $X^+, Q_1 \in P_1 \mathcal{B} P_1$;
- $\mathbf{s}(X^+) \perp Q_1$
- $D(P_1) = 2\big[D(\mathbf{s}(X^+)) + D(Q_1)\big]$;
- $q_{\mathcal{A}}(X^+) = \beta D(Q_1)$.

(II) Since $\mathbf{s}(X^-) = E_2$, it follows that:
- $X^-, Q_3 \in P_2 \mathcal{B} P_2$;
- $\mathbf{s}(X^-) \perp Q_3$
- $D(P_2) = 2\big[D(\mathbf{s}(X^-)) + D(Q_3)\big]$;
- $q_{\mathcal{A}}(X^-) = \beta D(Q_3)$.

We now use Theorem 4.1 to find

(I) a 2-folding $\Phi_1 = (A_1, B_1; V_1, W_1)$ of X^+ as βQ_1, with $\mathbf{s}(\Phi_1) \leq P_1$, and
(II) a 2-folding $\Gamma = (A_2, B_2; V_2, W_2)$ of X^- as βQ_2, with $\mathbf{s}(\Gamma) \leq P_2$.

It is trivial to see that the system $\Phi_2 = (-A_2, -B_; - V_2, -W_2)$ is a 2-folding of $-X^-$ as $-\beta Q_3$, again with $\mathbf{s}(\Phi_2) \leq P_2$. By Remark 4.1, the system

$$\Phi_1 + \Phi_2 = (A_1 - A_2, B_1 - B_2; V_1 - V_2, W_1 - W_2)$$

is a 2-folding of $X^+ - X^- = X$ as $\beta Q_1 - \beta Q_2$. Obviously the element $S = \beta Q_1 - \beta Q_2$ is spectrally symmetric. $\square$

Corollary 5.1. *If $\mathcal{A}$ is an AW*-factor, and if $X \in \mathcal{A}_{sa}$ is an element with $q_{\mathcal{A}}(X)$, satisfying the additional conditions* (i) *and* (ii) *from Theorem 5.1, then X can be written as a sum of three commuting spectrally symmetric elements in $\mathcal{A}$.*

Proof. Let S be a spectrally symmetric element, such that there exists a 2-folding $\Phi = (A_1, A_2; B_1, B_2)$ of X as S. Then

$$X = (A_1 - B_1) + (A_2 - B_2) + S$$

is a sum of the desired form. $\square$

The discussion from this point on is aimed at removing condition (ii) from the hypothesis of Theorem 5.1, and relaxing condition (i) as much as possible.

Lemma 5.1 (Small Packing)**.** *Let $\mathcal{A}$ be an AW^*-factor of type II_1, let $P \in \mathbf{P}(\mathcal{A})$ be a non-zero projection, and let $\mathcal{B}$ be an AW^*-subfactor of $P\mathcal{A}P$. For any element $X \in \mathcal{A}_{sa}$ with $X \perp P$, and any non-zero projection $Q \in \mathbf{P}(\mathcal{B})$, there exist five elements $A_1, A_2, B_1, B_2, Y \in \mathcal{A}_{sa}$, with the following properties*

(i) *A_1, A_2, B_1, B_2, Y all commute;*
(ii) *$A_1 \perp A_2$, $B_1 \perp B_2$, $A_1 \perp B_1$ and $A_2 \perp B_2$;*
(iii) *$A_1 \sim B_1$ and $A_2 \sim B_2$;*
(iv) *$A_1, A_2, B_1 \perp P$;*
(v) *$Y \in Q\mathcal{B}Q$;*
(vi) *$B_2P = PB_2 = Y$;*
(vii) *$X = A_1 + A_2 - B_1 - B_2 + Y$.*

Proof. We will assume $D_{\mathcal{A}}\big(\mathbf{s}(X)\big) > 0$ (the case $X = 0$ is trivial). Let $\lambda = D_{\mathcal{A}}(P)$, so that

$$q_{\mathcal{A}}(B) = \lambda q_{\mathcal{B}}(B),\ \ \forall\, B \in \mathcal{B}. \tag{57}$$

Let $\beta = D_{\mathcal{A}}(Q)$, so that $D_{\mathcal{B}}(Q) = \beta/\lambda$.

Fix some integer $n \geq 1$, such that

$$2n \geq \frac{D_{\mathcal{A}}\big(\mathbf{s}(X)\big)}{\beta},$$

and define the number

$$\alpha = \frac{D_{\mathcal{A}}\big(\mathbf{s}(X)\big)}{2n},$$

so that we have the equality $D_{\mathcal{A}}\big(\mathbf{s}(X)\big) = 2n\alpha$, and $D_{\mathcal{A}}(Q) \geq \alpha$.

Using Proposition 3.5 there is a scale $\mathcal{F} = \big(F, [0, 2n\alpha]\big)$ in $\mathcal{A}$ with $F(2n\alpha) = \mathbf{s}(X)$, and a function $f \in \mathfrak{R}[0, 2n\alpha]$, such that $X = \int_0^{2n\alpha} f\, d\mathcal{F}$. Fix also a full scale $\mathcal{G} = \big(G, [0, 1]\big)$ in $\mathcal{B}$, that contains Q, so that $D_{\mathcal{B}}\big(G(\beta/\lambda)\big) = Q$. By construction, one has

$$D_{\mathcal{A}}\big(G(t/\lambda)\big) = \lambda D_{\mathcal{B}}\big(G(t/\lambda)\big) = t,\ \ \forall\, t \in [0, \lambda]. \tag{58}$$

Since $\alpha \leq \beta$, we have $G(\alpha/\lambda) \leq Q$.

Using (58) it follows that the system $\mathcal{E} = \big(E, [0, (2n+1)\alpha]\big)$ defined by:

$$E(t) = \begin{cases} F(t) & \text{if } 0 \leq t \leq 2n\alpha \\ G\big((t - 2n\alpha)/\lambda\big) + F(2n\alpha) & \text{if } 2n\alpha < t < (2n+1)\alpha \end{cases}$$

is a scale in $\mathcal{A}$. Its key features are as follows:

(I) $X = \int_0^{2n\alpha} f\, d\mathcal{E}$;
(II) $E(t) \perp P$, $\forall\, t \in [0, 2n\alpha]$;
(III) $E(t) - E(2n\alpha) \in Q\mathcal{B}Q$, $\forall\, t \in [2n\alpha, (2n+1)\alpha]$.

Define the functions

$$f_k = f\big|_{[(k-1)\alpha,k\alpha]} \in \mathfrak{R}\big[(k-1)\alpha, k\alpha\big], \;\; k = 1,2,\dots,2n,$$

so that one has the equality

$$X = \sum_{k=1}^{2n} \int_{(k-1)\alpha}^{k\alpha} f_k \, d\mathcal{E}. \tag{59}$$

Define the sequence of functions $g_k \in \mathfrak{R}\big[(k-1)\alpha, k\alpha\big]$, $k = 1,2,\dots,2n$ starting with $g_1 = f_1$, and using the recursive formula

$$g_k = f_k + \Lambda_\alpha g_{k-1}, \;\; k = 2,3,\dots,2n.$$

Here $\Lambda_\alpha : \mathfrak{R}[a,b] \to \mathfrak{R}[a+\alpha, b+\alpha]$ denotes the translation map (see Remark 3.5).

Define now the sequences $(V_k)_{k=1}^{2n}$ and $(W_k)_{k=1}^{2n}$ by

$$V_k = \int_{(k-1)\alpha}^{k\alpha} g_k \, d\mathcal{E} \text{ and } W_k = \int_{k\alpha}^{(k+1)\alpha} \Lambda_\alpha g_k \, d\mathcal{E}, \;\; k = 1,2,\dots,2n.$$

The key features of these two sequences are described below.

Claim. The sequences $(V_k)_{k=1}^{2n}$ *and* $(W_k)_{k=1}^{2n}$ *have the following properties:*

(A) $V_1,\dots,V_{2n},W_1,\dots,W_{2n}$ *all commute;*
(B) $V_i \perp V_j$ *and* $W_i \perp W_j$, $\forall\, i,j \in \{1,\dots,2n\}$ *with* $i \neq j$*;*
(C) $V_i \perp P$, $\forall\, i \in \{1,\dots,2n\}$*;*
(D) $W_i \perp P$, $\forall\, i \in \{1,\dots,2n-1\}$*;*
(E) $V_i \perp W_j$, $\forall\, i,j \in \{1,\dots,2n\}$, *with* $j \neq i-1$.
(F) $W_{2n} \in Q\mathcal{B}Q$*;*
(G) $X = \sum_{i=1}^{2n} V_i - \sum_{j=1}^{2n-1} W_j$*;*
(A) $V_i \sim W_i$, $\forall\, i \in \{1,\dots,2n\}$.

The first assertion is trivial. To prove properties (A)-(E) we define the projections $R_k = E(k\alpha) - E((k-1)\alpha)$, $k = 1,\dots,2n+1$, and we observe that

- $\mathbf{s}(V_i) \leq R_i$ and $\mathbf{s}(W_i) \leq R_{i+1}$, $\forall\, i \in \{1,\dots,2n\}$;
- $R_i \perp R_j$, $\forall\, i,j \in \{1,\dots,2n+1\}$ with $i \neq j$;
- $R_i \perp P$, $\forall\, i \in \{1,\dots,2n\}$;
- $R_{2n+1} \in Q\mathcal{B}Q$

The fact that $W_{2n} = \int_{2n\alpha}^{(2n+1)\alpha} \Lambda_\alpha g_{2n} \, d\mathcal{E}$ belongs to $Q\mathcal{B}$ follows from condition (III) above.

Property (G) is quite obvious from (59), since $V_1 = \int_0^\alpha f_1 \, d\mathcal{E}$, and

$$V_k - W_{k-1} = \int_{(k-1)\alpha}^{k\alpha} (g_k - \Lambda_\alpha g_{k-1}) \, d\mathcal{E} = \int_{(k-1)\alpha}^{k\alpha} f_k \, d\mathcal{E}, \;\; \forall\, k \in \{2,3,\dots,2n\}.$$

Finally, property (H) is immediate from Corollary 3.3.

Having proven the above Claim, we now define the elements $A_1 = \sum_{k=1}^n V_{2k-1}$, $A_2 = \sum_{k=1}^n V_{2k}$, $B_1 = \sum_{j=1}^n W_{2k-1}$, $B_2 = \sum_{j=k}^n W_{2k}$, and $Y = W_{2n}$. Using the Claim and Corollary 2.1, it is pretty obvious that $A_1 \sim B_1$ and $A_2 \sim B_2$. The fact that the elements A_1, A_2, B_1, B_2, Y satisfy all the other desired conditions follows from the Claim. $\square$

Theorem 5.2. *Let $\mathcal{A}$ be an AW*factor of type* II_1*, and let $X \in \mathcal{A}_{sa}$ be an element with $q_{\mathcal{A}}(X) = 0$. If $D\big(\mathbf{s}(X)\big) < 1$, then X can be written as a sum $X = X_1+X_2+X_3$ of three commuting spectrally symmetric elements $X_1, X_2, X_3 \in \mathcal{A}_{sa}$.*

Proof. Put $P = I - \mathbf{s}(X)$, and denote the AW*-subalgebra $P\mathcal{A}P$ by $\mathcal{A}_0$. Of course $\mathcal{A}_0$ is an AW*-factor of type II_1. Fix some thick AW*-subfactor $\mathcal{B}$ of $\mathcal{A}_0$, of type II_1, as well as some projection $Q \in \mathbf{P}(\mathcal{B})$ with $D_{\mathcal{B}}(Q) < \frac{1}{2}$. Fix five elements $A_1, A_2, B_1, B_2, Y \in \mathcal{A}_{sa}$ that satisfy the conditions (i)-(vii) from Lemma 5.1.

Notice that using (i), (iii), and (vii), it follows that $q_{\mathcal{A}}(Y) = 0$. Let us concentrate for the moment on the element $Y \in \mathcal{A}_0$, which has $q_{\mathcal{A}_0}(Y) = 0$. On the one hand, Y belongs to the thick AW*-subfactor $\mathcal{B}$. On the other hand, by condition (v), we have $\mathbf{s}(Y) \leq Q$, so in particular we get $D_{\mathcal{B}}\big(\mathbf{s}(Y)\big) < \frac{1}{2}$. Using Theorem 5.1, there exists a spectrally symmetric element $S \in \mathcal{B}_{sa}$, and a 2-folding $\Phi = (Y_1, Y_2; S_1, S_2)$ in $\mathcal{A}_0$, with

$$Y = Y_1 + Y_2 \text{ and } S = S_1 + S_2. \tag{60}$$

Claim. The elements $A_1, A_2, B_1, B_2, Y_1, Y_2, S_1, S_2$ all commute. Moreover, one has the orthogonality relations

(A) *$A_1, A_2, B_1 \perp Y_1, Y_1, S_1, S_2$;*
(B) *$B_2 \perp S_1, S_2$.*

The relations (A) are clear, since $A_1, A_2, B_1 \perp P$, and $Y_1, Y_2, S_1, S_2 \in P\mathcal{A}P$. To prove the relations (B) we use the fact that $S_1, S_2 \in P\mathcal{A}P$, so that using condition (vi) from Lemma 5.1 and the fact that $\Phi = (Y_1, Y_2; S_1, S_2)$ is a folding, we have

$$B_2 S_k = B_2 P S_k = Y S_k = (Y_1 + Y_2) S_k = 0, \;\; k = 1, 2.$$

Bsed on these orthogonality relations, we see that the only commutation that needs to be checked is among B_2, Y_1, Y_2. Again using the fact that $Y_1, Y_2 \in P\mathcal{A}P$, and condition (vi) from Lemma 5.1, we have

$$B_2 Y_k = B_2 P Y_k = Y Y_k = Y_k Y = Y_k P B_2 = Y_k B_2, \;\; k = 1, 2,$$

and we are done.

Having proven the Claim, we now define the elements $X_1 = (A_1-B_1)+(Y_1-S_1)$, $X_2 = (A_2-B_2)+S$, and $X_3 = Y_2-S_2$. By the Claim, these three elements commute. Moreover, X_3 is obviously spectrally symmetric. Using the orthogonality relations from the Claim, combined with Lemma 5.1 and the features of the 2-folding Φ, we also have

- $A_1 - B_1$ and $Y_1 - S_1$ are spectrally symmetric, and $(A_1 - B_1) \perp (Y_1 - S_1)$,
- $A_2 - B_2$ and S are spectrally symmetric, and $(A_2 - B_2) \perp S$,

hence X_1 and X_2 are also spectrally symmetric.

Finally, using condition (vii) from Lemma 5.1 and (60) we have

$$X_1+X_2+X_3 = A_1-B_1+A_2-B_2+Y_1+Y_2-S_1-S_2+S = A_1-B_1+A_2-B_2+Y = X,$$

and we are done. ☐

Corollary 5.2. *Let $\mathcal{A}$ be an AW*-factor of type* II_1*, and let $X \in \mathcal{A}_{sa}$ be an element with $q_{\mathcal{A}}(X) = 0$. There exist three commuting spectrally symmetric elements $X_1, X_2, X_3 \in \mathrm{Mat}_2(\mathcal{A})$ – the 2×2 matrix algebra – such that*

$$X_1 + X_2 + X_3 = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}. \tag{61}$$

(According to Berberian's Theorem (see [1]), the matrix algebra $\mathrm{Mat}_2(\mathcal{A})$ is an AW*-factor of type II_1.)

Proof. Denote the matrix algebra $\mathrm{Mat}_2(\mathcal{A})$ by $\mathcal{A}_2$, and let $\tilde{X} \in \mathcal{A}_2$ denote the matrix in the right hand side of (61). It is obvious that, if we consider the projection

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then $\mathbf{s}(\tilde{X}) \leq E$. Since $D_{\mathcal{A}_2}(E) = \frac{1}{2} < 1$, and $q_{\mathcal{A}_2}(\tilde{X}) = \frac{1}{2} q_{\mathcal{A}}(X) = 0$, the desired conclusion follows imediately from Theorem 5.2. $\square$

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